

# Level Sets of the Takagi Function: Local Level Sets

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## ABSTRACT

The Takagi function  $\tau : [0, 1] \rightarrow [0, 1]$  is a continuous non-differentiable function constructed by Takagi in 1903. The level sets  $L(y) = \{x : \tau(x) = y\}$  of the Takagi function  $\tau(x)$  are studied by introducing a notion of local level set into which level sets are partitioned. Local level sets are simple to analyze, reducing questions to understanding the relation of level sets to local level sets, which is more complicated. It is known that for a “generic” full Lebesgue measure set of ordinates  $y$ , the level sets are finite sets. Here it is shown for a “generic” full Lebesgue measure set of abscissas  $x$ , the level set  $L(\tau(x))$  is uncountable. An interesting singular monotone function is constructed associated to local level sets, and is used to show the expected number of local level sets at a random level  $y$  is exactly  $\frac{3}{2}$ .

**Keywords** Binary expansion - Coarea formula - Hausdorff dimension - Level set - Singular function - Takagi function

**Mathematics Subject Classification** 26A27 - 26A45

## 1. Introduction

The Takagi function  $\tau(x)$  is a function defined on the unit interval  $x \in [0, 1]$  which was introduced by Takagi [33] in 1903 as an example of a continuous nondifferentiable function. It can be defined by

$$\tau(x) := \sum_{n=0}^{\infty} \frac{\langle\langle 2^n x \rangle\rangle}{2^n} \quad (1.1)$$

where  $\langle\langle x \rangle\rangle := \inf_{n \in \mathbb{Z}} |x - n|$  is the distance from  $x$  to the nearest integer. Variants of this function were presented by van der Waerden [35] in 1930 and de Rham [29] in 1957.

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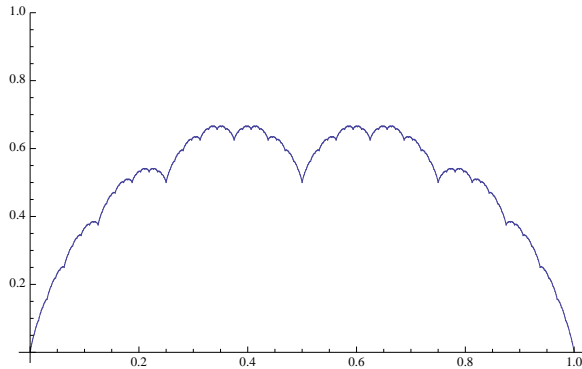


Figure 1: Graph of the Takagi function  $\tau(x)$ .

An alternate interpretation of the Takagi function involves the symmetric tent map  $T : [0, 1] \rightarrow [0, 1]$ , given by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} \quad (1.2)$$

(see [19] for further references). Then we have

$$\tau(x) := \frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{1}{2^n} T^{(n)}(x) \right),$$

where  $T^{(n)}(x)$  denotes the  $n$ -th iterate of  $T(x)$ . The Takagi function appears in many contexts and has been studied extensively; see the recent surveys of Allaart and Kawamura [5] and of the first author [24].

In this paper we consider certain properties of the graph of the Takagi function

$$\mathcal{G}(\tau) := \{(x, \tau(x)) : 0 \leq x \leq 1\},$$

which is pictured in Figure 1. It is well known that the values of the Takagi function satisfy  $0 \leq \tau(x) \leq \frac{2}{3}$ . It is also known that this graph has Hausdorff dimension 1 in  $\mathbb{R}^2$ , see Mauldin and Williams [28, Theorem 7]), and furthermore it is  $\sigma$ -finite, see Anderson and Pitt [7, Thm. 6.4]. Here we study the structure of the level sets of this graph. We make the following definition, which contains a special convention concerning dyadic rationals which simplifies theorem statements.

**Definition 1.1.** For  $0 \leq y \leq \frac{2}{3}$  the (*global*) *level set*  $L(y)$  at level  $y$  is

$$L(y) := \{x : \tau(x) = y, \ 0 \leq x \leq 1\}.$$

We make the convention that  $x$  specifies a binary expansion; thus each dyadic rational value  $x = \frac{m}{2^n}$  in a level set occurs twice, labeled by its two possible binary expansions. (Technically  $L(y)$  is a multiset, with multiplicities 1 or 2.)

Level sets have a complicated structure, depending on the value of  $y$ . It is known that there are different levels  $y$  where the level set  $L(y)$  is finite, countably infinite, or uncountably infinite, respectively. In 1959 Kahane [21, Sec. 1] noted that the level set  $L(\frac{2}{3})$  was a perfect,

totally disconnected set of Lebesgue measure 0, and in 1984 Baba [8] showed that  $L(\frac{2}{3})$  has Hausdorff dimension  $\frac{1}{2}$ . The second author recently proved ([27]) that the Hausdorff dimension of any level set is at most 0.668, and conjectured that the example of Baba achieves the largest possible Hausdorff dimension. Recently de Amo, Bhouiri, Díaz Carrillo and Fernández-Sánchez [6] proved this conjecture. The level sets at a rational level  $y \in \mathbb{Q}$  are particularly interesting. Knuth [23, Sect. 7.2.1.3, Exercises 82-85] gave a (not necessarily halting) algorithm to determine the structure of level sets at rational levels, revealing very complicated behaviors. For example he determined that  $L(\frac{1}{5})$  has a finite level set with exactly two elements, namely  $L(\frac{1}{5}) = \{x, 1-x\}$  with  $x = \frac{83581}{87040}$ . He also noted that  $L(\frac{1}{2})$  is countably infinite; we include a proof in Theorem 7.1 below. In 2008 Buczolic [13] proved that, in the sense of Lebesgue measure on  $y \in [0, \frac{2}{3}]$ , almost all level sets  $L(y)$  are finite sets.

The object of this paper is to introduce and study the notion of “local level set”. These are sets determined locally by combinatorial operations on the binary expansion of a real number  $x$ ; they are closed sets and we show that each level set decomposes into a disjoint union of local level sets. (The convention on dyadic rationals made in the definition above is needed for disjointness of the union.) The structure of local level sets is completely analyzable: they are either finite sets or Cantor sets. Information about the Hausdorff dimension of such sets can readily be deduced from properties of the binary expansion of  $x$ .

We then study the relation of local level sets and level sets. How many local level sets are there in a given level set? To approach this question, we study the behavior of the Takagi function restricted to the set  $\Omega^L$  of left hand (abscissa) endpoints  $x$  of all the local level sets; these endpoints parametrize the totality of all local level sets. We show that  $\Omega^L$  is a closed perfect set (Cantor set) which has Lebesgue measure 0; in a sequel ([25]) we show it has Hausdorff dimension 1. We show the Takagi function behaves relatively nicely when restricted to  $\Omega^L$ , namely that  $\tau^S(x) := \tau(x) + x$  is a monotone singular continuous function on this set. It is therefore the integral of a singular probability measure on  $[0, 1]$ , which we call the Takagi singular measure. Using this function we deduce that the expected number of local level sets at a random level  $0 \leq y \leq \frac{2}{3}$  is finite, and we determine that this expected value is exactly  $\frac{3}{2}$ . We also show that there is a dense set of values  $y$  having an infinite number of distinct local level sets.

Local level sets provide a way to take apart level sets and better understand their structure. In the rest of the introduction we state the main results of this paper in more detail.

### 1.1. Local level sets

This notion of local level set is attached to the binary expansion of abscissa point  $x \in [0, 1]$ . We show that certain combinatorial flipping operations applied to the binary expansion of  $x$  yield new points  $x'$  in the same level set. The totality of points reachable from  $x$  by these combinatorial operations will comprise the local level set  $L_x^{loc}$  associated to  $x$ .

To describe this, let  $x \in [0, 1]$  have a binary expansion:

$$x := \sum_{j=1}^{\infty} \frac{b_j}{2^j} = 0.b_1b_2b_3\dots, \quad \text{each } b_j \in \{0, 1\}.$$

The *flip operation* (or *complementing operation*) on a single binary digit  $b$  is

$$\bar{b} := 1 - b.$$

We associate to the binary expansion the *digit sum function*  $N^1(x)$  given by

$$N_j^1(x) := b_1 + b_2 + \cdots + b_j.$$

We also associate to the binary expansion the *deficient digit function*  $D_j(x)$  given by

$$D_j(x) := j - 2N_j^1(x) = j - 2(b_1 + b_2 + \cdots + b_j).$$

Here  $D_j(x)$  counts the excess of binary digits  $b_k = 0$  over those with  $b_k = 1$  in the first  $j$  digits, i.e. it is positive if there are more 0's than 1's in the first  $j$  digits. Note that for dyadic rationals  $x = \frac{m}{2^n}$  the function values depend on which binary expansion is used.

We next associate to any  $x$  the sequence of digit positions  $j$  at which tie-values  $D_j(x) = 0$  occur, which we call *balance points*; note that all such  $j$  are even. The *balance-set*  $Z(x)$  associated to  $x$  is the set of balance points, and is denoted

$$Z(x) := \{c_k : D_{c_k}(x) = 0\}. \quad (1.3)$$

where we define  $c_0 = c_0(x) = 0$  and set  $c_0(x) < c_1(x) < c_2(x) < \dots$ . This sequence of tie-values may be finite or infinite. If it is finite, ending in  $c_n(x)$ , we make the convention to adjoin a final “balance point”  $c_{n+1}(x) = +\infty$ . We call a “*block*” an indexed set of digits between two consecutive balance points,

$$B_k(x) := \{b_j : c_k(x) < j \leq c_{k+1}(x)\},$$

which includes the second balance point but not the first. We define an equivalence relation on blocks, written  $B_k(x) \sim B_{k'}(x')$  to mean the block endpoints agree ( $c_k(x) = c_{k'}(x')$  and  $c_{k+1}(x) = c_{k'+1}(x')$ ) and either  $B_k(x) = B_{k'}(x')$  or  $B_k(x) = \bar{B}_{k'}(x')$ , where the bar operation flips all the digits in the block, i.e.

$$b_j \mapsto \bar{b}_j := 1 - b_j, \quad c_k < j \leq c_{k+1}.$$

Finally, we define an equivalence relation  $x \sim x'$  to mean that they have identical balance-sets  $Z(x) \equiv Z(x')$ , and furthermore every block  $B_k(x) \sim B_k(x')$  for  $k \geq 0$ . Note that  $x \sim 1 - x$ ; this corresponds to a flipping operation being applied to every binary digit. We will show (Theorem 3.1) that the equivalence relation  $x \sim x'$  implies that  $\tau(x) = \tau(x')$  so that  $x$  and  $x'$  are in the same level set of the Takagi function.

**Definition 1.2.** The *local level set*  $L_x^{loc}$  associated to  $x$  is the set of equivalent points,

$$L_x^{loc} := \{x' : x' \sim x\}.$$

We use again the convention that  $x$  and  $x'$  denote binary expansions, and hence dyadic rational numbers are represented by two distinct binary expansions.

Each local level set  $L_x^{loc}$  is a closed set. It is a finite set if the balance-set  $Z(x)$  is finite, and is a Cantor set (perfect totally disconnected set) if  $Z(x)$  is infinite. Note that if  $x$  is the expansion of a dyadic rational, then  $L_x^{loc}$  is finite and consists entirely of (expansions of) dyadic rationals.

**Theorem 1.3.** (Local level set partition)

- (1) Each local level set  $L_x^{loc}$  is a closed set contained in some level set.
- (2) Two local level sets  $L_x^{loc}$  and  $L_{x'}^{loc}$  either coincide or are disjoint. Thus each level set  $L(y)$  partitions into a disjoint union of local level sets.

This easy result is proved as part of Theorem 3.1 in Sect. 3.1. A priori this disjoint union could be finite, countable or uncountable. In Theorem 7.1 we give an example of a level set that is a countably infinite union of local level sets; for this case  $y = \frac{1}{2}$  is a dyadic rational.

The Hausdorff dimension of a local level set  $L_x^{loc}$  is restricted by the nature of its balance-set  $Z(x)$ . A necessary condition to have positive Hausdorff dimension is that  $Z(x)$  must have positive upper asymptotic density in  $\mathbb{N}$ . This allows us to deduce the following result.

**Theorem 1.4.** (Generic local level sets) *For a full Lebesgue measure set of abscissa points  $x$  the local level set  $L_x^{loc}$  is a Cantor set (closed totally disconnected perfect set) having Hausdorff dimension 0.*

Theorem 1.4 is proved in Sect. 3.2. This result implies that if an abscissa value  $x$  is picked at random in  $[0, 1]$ , then with probability one the level set  $L(\tau(x))$  is uncountably infinite. This result differs strikingly from that of Buczolic [13], who showed that if an ordinate value  $y$  is picked uniformly in  $[0, \frac{2}{3}]$  then the level set  $L(y)$  is finite with probability one. There is no inherent contradiction here: drawing an abscissa value  $x$  will preferentially select levels whose level set  $L(\tau(x))$  is “large”, and Theorem 1.4 quantifies “large.”

In Sect. 3.3 we completely determine the structure of local level sets that contain a rational number. We prove they are either a finite set or a Cantor set of positive Hausdorff dimension, and characterize when each case occurs (Theorem 3.3). One can check directly that  $x_0 = \frac{1}{3}$  has a local level set of Hausdorff dimension  $\frac{1}{2}$  (at level  $y = \frac{2}{3}$ ), which shows that the Hausdorff dimension upper bound of  $\frac{1}{2}$  for level sets obtained in [6] is sharp also for local level sets.

## 1.2. Expected number of local level sets

Our second object is to relate local level sets to level sets. How many local level sets belong to a given level set? To approach this problem we first study (in Sect. 4) the set  $\Omega^L$  of all left hand endpoints of local level sets, because this set parameterizes all the local level sets. In Theorem 4.6 we characterize its members in terms of their binary expansions: they are exactly the values  $x$  with binary expansions such that

$$D_j(x) \geq 0 \quad \text{for all } j \geq 1.$$

We call this latter set the *deficient digit set* and show it is a closed, perfect set (Cantor set) of Lebesgue measure zero.

In §5 we define a new function, the *flattened Takagi function*  $\tau^L(x)$ , which agrees with  $\tau(x)$  on  $\Omega^L$  and is defined by linear interpolation across the gaps removed in constructing  $\Omega^L$ . We prove  $\tau^L(x)$  to be of bounded variation, and determine a Jordan decomposition. This consists of a nondecreasing piece  $F_+(x) := \tau^L(x) + x$  which we establish is a singular function whose points of increase are supported on  $\Omega^L$ , and a strictly decreasing piece  $F_-(x) := -x$  which is absolutely continuous. We name the function

$$\tau^S(x) := F_+(x) = \tau^L(x) + x$$

the *Takagi singular function*, based on the following result, which is needed in the proof that the flattened Takagi function has bounded variation.

**Theorem 1.5.** (Takagi singular function) *The function  $\tau^S(x)$  defined by  $\tau^S(x) = \tau(x) + x$  for  $x \in \Omega^L$  is a nondecreasing function on  $\Omega^L$ . Define its extension to all  $x \in [0, 1]$  by*

$$\tau^S(x) := \sup\{\tau^S(x_1) : x_1 \leq x \text{ with } x_1 \in \Omega^L\}.$$

*Then the function  $\tau^S(x)$  is a monotone singular function. That is, it is a nondecreasing continuous function having  $\tau^S(0) = 0, \tau^S(1) = 1$ , which has derivative zero at (Lebesgue) almost all points of  $[0, 1]$ . The closure of the set of points of increase of  $\tau^S(x)$  is the deficient digit set  $\Omega^L$ .*

In a sequel ([25]) we study a nonnegative Radon measure  $d\mu_S$ , called the *Takagi singular measure*, such that

$$\tau^S(x) = \int_0^x d\mu_S,$$

which is a probability measure on  $[0, 1]$ . This measure is singular with respect to Lebesgue measure. There we show that its support  $\text{Supp}(\mu_S) = \Omega^L$  has (full) Hausdorff dimension 1. The Takagi singular measure is not translation-invariant, but it has certain self-similarity properties under dyadic rescalings. These are useful in explicitly computing the measure of various interesting subsets of  $\Omega^L$ . One may compare analogous properties of the Cantor function, see Dovghoshey et al [14, Sect. 5].

The bounded variation property of the flattened Takagi function is used to count the average number of local level sets, as follows.

**Theorem 1.6.** (Expected number of local level sets) *With respect to uniform (Lebesgue) measure on the ordinate space  $[0, \frac{2}{3}]$  a full measure set of points have a finite number of local level sets. Furthermore the expected number of local level sets on a given level  $y \in [0, \frac{2}{3}]$  is  $\frac{3}{2}$ .*

This result is proved as Theorem 6.3, using the coarea formula for functions of bounded variation. We show that this result is non-trivial in that there are infinitely many levels containing infinitely many distinct local level sets.

**Theorem 1.7.** (Infinite Number of Local Level Sets) *There exists a dense set of ordinate values  $y$  in  $[0, \frac{2}{3}]$ , which are all dyadic rationals, such that the level set  $L(y)$  contains an infinite number of distinct local level sets.*

This theorem follows directly from a result proved in Sect. 7 (Theorem 7.2). This in turn is derived from the fact that  $L(\frac{1}{2})$  is a countable set which contains a countably infinite number of local level sets (Theorem 7.1).

In the final Sect. 8 we formulate some open questions suggested by this work.

### 1.3. Extensions of results and related work

The Takagi function has self-affine properties, and there has been extensive study of various classes of self-affine functions. In particular, in the late 1980's Bertoin [9], [10] studied the Hausdorff dimension of level sets of certain classes of self-affine functions; however his results do not cover the Takagi function.

In 1997 Yamaguti, Hata and Kigami [36, Chap. 3] gave a general definition of a family  $F(t, x)$  of Takagi-like functions depending on a parameter  $0 < t < 1$  as follows: let  $g(x)$  be a bounded measurable function defined on  $[0, 1]$  and let  $\Phi : [0, 1] \rightarrow [0, 1]$  be a continuous mapping, then set

$$F(t, x) := \sum_{n=0}^{\infty} t^n g(\Phi^n(x)).$$

If we specialize to take  $\Phi(x) = 2g(x) = T(x)$ , the tent map in (1.2), then the parameter value  $t = \frac{1}{2}$  gives the Takagi function

$$F\left(\frac{1}{2}, x\right) = 2 \left( \sum_{n=1}^{\infty} \frac{1}{2^n} T^n(x) \right) = \tau(x).$$

If one now changes the parameter value to  $t = \frac{1}{4}$ , then one gets instead ([36, p. 35]) the smooth function

$$F\left(\frac{1}{4}, x\right) = \frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{1}{4^n} T^n(x) \right) = \frac{1}{2} x(1 - x).$$

The level sets of this function are finite. These examples show an extreme dependence of level set structure on the parameter  $t$ . Our analysis uses the piecewise linear nature of the function  $\Phi(x)$  in a strong way, and also uses specific properties of the geometric scaling by the parameter  $t$  at value  $t = \frac{1}{2}$ .

The methods presented should extend to various functions similar in construction to the Takagi function, such as van der Waerden's function ([35]). They also extend to intersections of the graph of the Takagi function with parallel families of lines having integer slope, a device used by the second author [27]. In this paper we have treated only Hausdorff dimension, while the paper [27] also obtained upper bounds for Minkowski dimension (a.k.a. box counting dimension) of level sets. Some results of this paper (e.g. Theorem 1.4) may be strengthened to give Minkowski dimension upper bounds.

In [25] we further analyze the structure of global level sets  $L(y)$  using local level sets. We give a new proof of a theorem of Buczolic [13] showing that if one draws  $y$  uniformly from  $[0, \frac{2}{3}]$ , then with probability one the level set  $L(y)$  is a finite set; we improve on it by showing that the expected number of points in such a “random” level set  $L(y)$  is infinite. We also complement this result by showing that the set of levels  $y$  having a level set of positive Hausdorff dimension is “large” in the sense that it has full Hausdorff dimension 1, although it is of Lebesgue measure 0.

Subsequent to this paper, Allaart [1] [2] obtains many further results on local level sets. He shows that dyadic ordinates  $y = \frac{k}{2^n}$  have finite or countable level sets, and he determines information on cardinalities of finite level sets.

Finally we remark that there has been much study of the non-differentiable nature of the Takagi function in various directions, see for example Allaart and Kawamura ([3], [4]) and references therein. It is considered as an example in Tricot [34, Section 6].

## 2. Basic Properties of the Takagi Function

We recall some basic facts and include proofs for the reader's convenience. We first give Takagi's formula for his function, which assigns a value  $\tau(x)$  directly to a binary expansion of

$x = 0.b_1b_2b_3\dots$ . Dyadic rationals  $\frac{k}{2^n}$  have two distinct binary expansions, and one checks the assigned value  $\tau(x)$  is the same for both expansions. For  $0 \leq x \leq 1$  the distance to the nearest integer function  $\langle\langle x \rangle\rangle$  is

$$\langle\langle x \rangle\rangle := \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2}, \text{ i.e. } b_1 = 0 \\ 1 - x & \text{if } \frac{1}{2} \leq x \leq 1, \text{ i.e. } b_1 = 1. \end{cases}$$

For  $n \geq 0$ , we have

$$\langle\langle 2^n x \rangle\rangle = \begin{cases} 0.b_{n+1}b_{n+2}b_{n+3}\dots & \text{if } b_{n+1} = 0 \\ 0.\bar{b}_{n+1}\bar{b}_{n+2}\bar{b}_{n+3}\dots & \text{if } b_{n+1} = 1, \end{cases} \quad (2.1)$$

where we use the bar-notation

$$\bar{b} = 1 - b, \quad \text{for } b = 0 \text{ or } 1,$$

to mean complementing a bit.

**Lemma 2.1.** (Takagi [33]) *For  $x = 0.b_1b_2b_3\dots$  the Takagi function is given by*

$$\tau(x) = \sum_{m=1}^{\infty} \frac{\ell_m}{2^m}, \quad (2.2)$$

in which  $0 \leq \ell_m = \ell_m(x) \leq m - 1$  is the integer

$$\ell_m(x) = \#\{i : 1 \leq i < m, \ b_i \neq b_m\}.$$

In terms of the digit sum function  $N_m^1(x) = b_1 + b_2 + \dots + b_m$ ,

$$\ell_{m+1}(x) = \begin{cases} N_m^1(x) & \text{if } b_{m+1} = 0, \\ m - N_m^1(x) & \text{if } b_{m+1} = 1. \end{cases} \quad (2.3)$$

**Proof.** From the definition

$$\tau(x) = \sum_{n=0}^{\infty} \frac{\langle\langle 2^n x \rangle\rangle}{2^n}$$

Now (2.1) gives

$$\frac{\langle\langle 2^n x \rangle\rangle}{2^n} = \begin{cases} \sum_{j=1}^{\infty} \frac{b_{n+j}}{2^{n+j}} & \text{if } b_{n+1} = 0, \\ \sum_{j=1}^{\infty} \frac{\bar{b}_{n+j}}{2^{n+j}} & \text{if } b_{n+1} = 1. \end{cases}$$

We substitute this into the formula for  $\tau(x)$  and collect all terms having a given denominator  $\frac{1}{2^m}$ , coming from  $m = n + j$  with  $1 \leq j \leq m$ . For  $m = n + j$  we get a contribution of  $\frac{1}{2^m}$  whenever  $b_{n+j} := b_m = 1$  if  $b_{n+1} = 0$ , and whenever  $b_{n+j} := b_m = 0$  if  $b_{n+1} = 1$ , otherwise get 0 contribution. Adding up over  $j$ , we find the total contribution is  $\frac{\ell_m}{2^m}$  where  $\ell_m(x)$  counts the number of  $b_j$ ,  $1 \leq j < m$  having the opposite parity to  $b_m$ , which is (2.2). Note that  $\ell_1(x) \equiv 0$ , so the summation (2.2) really starts with  $m = 2$ . The formulas (2.3) follow by inspection; note that  $m - N_m^1(x) = N_m^1(1 - x)$  (making an appropriate convention for dyadic rationals).  $\square$

We next recall two basic functional equations, see Kairies, Darslow and Frank [22].



**Lemma 2.2.** (Takagi functional equations)

The Takagi function satisfies two functional equations, each valid for  $0 \leq x \leq 1$ . These are the reflection equation

$$\tau(x) = \tau(1 - x), \quad (2.4)$$

and the dyadic self-similarity equation

$$2\tau\left(\frac{x}{2}\right) = \tau(x) + x. \quad (2.5)$$

**Proof.** Here (2.4) follows directly from (1.1), since  $\langle\langle kx \rangle\rangle = \langle\langle k(1-x) \rangle\rangle$  for  $k \in \mathbb{Z}$ . To obtain (2.5), let  $x = 0.b_1b_2b_3\dots$  and set  $y := \frac{x}{2} = 0.0b_1b_2b_3\dots$ . Then  $\langle\langle y \rangle\rangle = y$ , whence (1.1) gives

$$2\tau(y) = 2\langle\langle y \rangle\rangle + 2\left(\sum_{n=1}^{\infty} \frac{\langle\langle 2^n y \rangle\rangle}{2^n}\right) = x + \sum_{m=0}^{\infty} \frac{\langle\langle 2^m x \rangle\rangle}{2^m} = x + \tau(x). \quad \square$$

We note that the Takagi function can be characterized as the unique continuous function on  $[0, 1]$  satisfying these two functional equations (Knuth [23, Exercise 82, solution p. 740]).

Next we recall the construction of the Takagi function  $\tau(x)$  as a limit of piecewise linear approximations

$$\tau_n(x) := \sum_{j=0}^{n-1} \frac{\langle\langle 2^j x \rangle\rangle}{2^j},$$

which we name the *partial Takagi function* of level  $n$ . We require some notation concerning the binary expansion:

$$x = \sum_{j=1}^{\infty} \frac{b_j}{2^j} = 0.b_1b_2b_3\dots, \quad \text{each } b_j \in \{0, 1\}.$$

**Definition 2.3.** Let  $x \in [0, 1]$  have binary expansion  $x = \sum_{j=1}^{\infty} \frac{b_j}{2^j} = 0.b_1b_2b_3\dots$ , with each  $b_j \in \{0, 1\}$ . For each  $j \geq 1$  we define the following integer-valued functions.

(1) The *digit sum function*  $N_j^1(x)$  is

$$N_j^1(x) := b_1 + b_2 + \dots + b_j.$$

We also let  $N_j^0(x) = j - N_j^1(x)$  count the number of 0's in the first  $j$  binary digits of  $x$ .

(2) The *deficient digit function*  $D_j(x)$  is given by

$$D_j(x) := N_j^0(x) - N_j^1(x) = j - 2N_j^1(x) = j - 2(b_1 + b_2 + \dots + b_j). \quad (2.6)$$

Here we use the convention that  $x$  denotes a binary expansion; dyadic rationals have two different binary expansions, and all functions  $N_j^0(x)$ ,  $N_j^1(x)$ ,  $D_j(x)$  depend on which binary expansion is used.

The name “deficient digit function” reflects the fact that  $D_j(x)$  counts the excess of binary digits  $b_k = 0$  over those with  $b_k = 1$  in the first  $j$  digits, i.e. it is positive if there are more 0's than 1's.

**Lemma 2.4.** (Piecewise linear approximations to Takagi function)

The piecewise linear function  $\tau_n(x) = \sum_{j=0}^{n-1} \frac{\langle 2^j x \rangle}{2^j}$  is linear on each dyadic interval  $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ .

(1) On each such interval  $\tau_n(x)$  has integer slope between  $-n$  and  $n$  given by the deficient digit function

$$D_n(x) = N_n^0(x) - N_n^1(x) = n - 2(b_1 + b_2 + \cdots + b_n),$$

Here  $x = 0.b_1b_2b_3\ldots$  may be any interior point on the dyadic interval, and can also be an endpoint provided the dyadic expansion ending in 0's is taken at the left endpoint  $\frac{k}{2^n}$  and that ending in 1's is taken at the right endpoint  $\frac{k+1}{2^n}$ .

(2) The values  $\{\tau_n(x) : n \geq 1\}$  converge uniformly to  $\tau(x)$ , with

$$|\tau_n(x) - \tau(x)| \leq \frac{2}{3} \cdot \frac{1}{2^n}. \quad (2.7)$$

The functions  $\tau_n(x)$  approximate the Takagi function monotonically from below

$$\tau_1(x) \leq \tau_2(x) \leq \tau_3(x) \leq \dots$$

For a dyadic rational  $x = \frac{k}{2^n}$ , perfect approximation occurs at the  $n$ -th step with

$$\tau(x) = \tau_m(x), \quad \text{for all } m \geq n.$$

**Proof.** All statements follow easily from the observation that each function  $f_n(x) := \frac{\langle 2^n x \rangle}{2^n}$  is a piecewise linear sawtooth function, linear on dyadic intervals  $[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}]$ , with slope having value  $+1$  if the binary expansion of  $x$  has  $b_{n+1} = 0$  and slope having value  $-1$  if  $b_{n+1} = 1$ . The inequality in (2.7) also uses the fact that  $\max_{x \in [0,1]} \tau(x) = \frac{2}{3}$ .  $\square$

The Takagi function itself can be directly expressed in terms of the deficient digit function. The relation (2.6) compared with the definition (2.3) of  $\ell_m(x)$  yields

$$\ell_{m+1}(x) = \frac{m}{2} - \frac{1}{2}(-1)^{b_{m+1}} D_m(x).$$

Substituting this in Takagi's formula (2.2) and simplifying yields the formula

$$\tau(x) = \frac{1}{2} - \frac{1}{4} \left( \sum_{m=0}^{\infty} (-1)^{b_{m+1}} \frac{D_m(x)}{2^m} \right).$$

We conclude this section with a self-similarity property of the Takagi function associated to dyadic rationals  $x = \frac{k}{2^n}$ .

**Lemma 2.5.** (Takagi self-affinity)

For an arbitrary dyadic rational  $x_0 = \frac{k}{2^n}$  then for  $x \in [\frac{k}{2^n}, \frac{k+1}{2^n}]$  given by  $x = x_0 + \frac{w}{2^n}$  with  $w \in [0, 1]$ ,

$$\tau(x) = \tau(x_0) + \frac{1}{2^n}(\tau(w) + D_n(x_0)w). \quad (2.8)$$

That is, the graph of  $\tau(x)$  on  $[\frac{k}{2^n}, \frac{k+1}{2^n}]$  is a miniature version of the tilted Takagi function  $\tau(x) + D_n(x_0)x$ , vertically shifted by  $\tau(x_0)$ , and shrunk by a factor  $\frac{1}{2^n}$ .

**Proof.** By Lemma 2.4(1), we have  $\tau_n(x_0 + \frac{w}{2^n}) = \tau_n(x_0) + D_n(x_0) \cdot \frac{w}{2^n}$ . Therefore, by (1.1) it follows that

$$\begin{aligned}\tau(x) &= \tau_n(x) + \sum_{j=n}^{\infty} \frac{\langle\langle 2^j x \rangle\rangle}{2^k} \\ &= \tau_n(x_0) + D_n(x_0) \cdot \frac{w}{2^n} + \sum_{j=n}^{\infty} \frac{\langle\langle 2^j (\frac{w}{2^n}) \rangle\rangle}{2^j} \\ &= \tau(x_0) + \frac{1}{2^n}(\tau(w) + D_n(x_0)w). \quad \square.\end{aligned}$$

**Remark.** Lemma 2.5 simplifies in the special case that  $D_n(x_0) = 0$ , in which case we call  $x_0$  a *balanced dyadic rational*; this can only occur when  $n = 2m$  is even. The formula (2.8) becomes

$$\tau(x) = \tau(x_0) + \frac{\tau(w)}{2^n},$$

which shows the graph of the Takagi function over the subinterval  $[\frac{k}{2^n}, \frac{k+1}{2^n}] \subseteq [0, 1]$ , up to a translation, consists of the image of the entire graph scaled by  $\frac{1}{2^n}$ . Balanced dyadic rationals play a special role in our analysis, see Definition 4.3.

### 3. Properties of Local Level Sets

In this section we derive some basic properties of local level sets. Then we determine the size of “abscissa generic” local level sets, showing these are uncountable sets of Hausdorff dimension 0. Finally we determine the structure of local level sets that contain a rational number  $x$ , showing they are either finite sets or Cantor sets of positive Hausdorff dimension.

#### 3.1. Partition into local level sets

We first show that level sets partition into local level sets.

**Theorem 3.1.** (1) *Local level sets  $L_x^{loc}$  are closed sets. Two local level sets either coincide or are disjoint.*

(2) *Each local level set  $L_x^{loc}$  is contained in a level set:  $L_x^{loc} \subseteq L(\tau(x))$ . That is, if  $x_1 \sim x_2$  then  $\tau(x_1) = \tau(x_2)$ .*

(3) *Each level set  $L(y)$  partitions into local level sets*

$$L(y) = \bigcup_{\substack{x \in \Omega^L \\ \tau(x)=y}} L_x^{loc}. \quad (3.1)$$

Here  $\Omega^L$  denotes the collection of leftmost endpoints of all local level sets.

**Proof.** (1) A local level set, specified by a binary expansion of  $x$ , is generated by any one of its elements, by block-flipping operations (allowing an infinite number of blocks to be flipped at once) including  $x \rightarrow 1 - x$ : if  $x_2 \in L_{x_1}^{loc}$  then  $x_1 \in L_{x_2}^{loc}$  and  $L_{x_1}^{loc} = L_{x_2}^{loc}$ . Each set  $L_x^{loc}$  is closed, because any Cauchy sequence  $\{x_n : n \geq 1\}$  in  $L_x^{loc}$  must eventually “freeze” the choice made in any finite initial set of “blocks”, so that for each  $k \geq 1$  there is a value  $n(k)$  so that  $B_k(x_n) = B_k(x_m)$  whenever  $n, m \geq n(k)$ . Since all balance-sets  $Z(x_n)$  coincide, the limit value

$x_\infty$  has  $B_k(x_\infty) = B_k(x_n)$  for all  $n \geq n(k)$  so  $B_k(x_\infty) \sim B_k(x)$  for all  $k \geq 0$ . This same relation shows that the first  $k$  balance points of  $x_\infty$  coincide with those of such  $x_n$ , and letting  $k \rightarrow \infty$  we have  $Z(x_\infty) = Z(x)$ . Thus  $x_\infty \sim x$ , so  $x_\infty \in L_x^{loc}$ .

(2) We assert that if  $x \sim x'$ , with  $x'$  obtained from  $x$  by flipping a single block of symbols, then  $\tau(x) = \tau(x')$ . This holds since a block-flip after the  $k$ th binary digit of  $x$  corresponds to a reflection of  $x$  about the center of the dyadic interval of length  $\frac{1}{2^k}$  containing  $x$ ; by Lemma 2.5 and Lemma 2.2, the Takagi function restricted to this interval has this reflection symmetry. The case  $\tau(x) = \tau(x'')$  of general  $x''$  in  $L_x^{loc}$  will then follow by flipping the blocks of  $x$  in increasing order as necessary to match those of  $x''$ , getting a sequence  $\{x_n : n \geq 1\} \subset L_x^{loc}$  with  $\tau(x_n) = \tau(x)$ . Now  $\lim_{n \rightarrow \infty} x_n = x''$  so using the fact that  $\tau(x)$  is a continuous function, we conclude  $\tau(x'') = \lim_{n \rightarrow \infty} \tau(x_n) = \tau(x)$ .

(3) Local level sets, being closed, have a leftmost endpoint, and we can then uniquely label local level sets with their leftmost endpoint.  $\square$

We immediately deduce that any level set that is countably infinite must contain infinitely many local level sets.

**Corollary 3.2.** (Countable Level Sets) (1) *Each local level set  $L_x^{loc}$  is either finite or uncountable.*

(2) *Each level set  $L_x^{loc}$  that is countably infinite is necessarily a countable disjoint union of finite local level sets.*

**Proof.** (1) This dichotomy for  $L_x^{loc}$  is determined by whether the balance set  $Z(x)$  is finite or infinite, since  $L_x^{loc}$  is a Cantor set in the latter case.

(2) If  $L(y)$  is countably infinite, all local level sets it contains must be finite, by (1). Thus there must be infinitely many of them.  $\square$

We will later show that case (2) above occurs: Theorem 7.1 proves that  $L(\frac{1}{2})$  is countably infinite.

### 3.2. Generic local level sets

We analyze the size of “abscissa generic” local level sets sampled by choosing  $x$  uniformly on  $[0, 1]$ , and prove these are Cantor sets of Hausdorff dimension 0.

**Proof of Theorem 1.4.** The sequence of binary digits in  $x = 0.b_1b_2b_3\dots$  of a real number in  $[0, 1]$  corresponds to taking a (random) walk on the integer lattice, starting at point  $s_0 = 0$ , with steps  $+1$  or  $-1$ , with  $b_j = 0$  corresponding to taking a step in the positive direction, and  $b_j = 1$  corresponding to taking a step in the negative direction, i.e. at time  $k$  the walk is at position

$$s_k = s_0 + \sum_{j=1}^k (-1)^{b_j}.$$

The interval  $[0, 1]$  sampled by drawing a random point  $x$  with the uniform distribution (i.e. using Lebesgue measure on  $[0, 1]$ ) corresponds in probability to taking a simple random walk with equal probability steps. (See Billingsley [11, Sect. 3], [12]).

The relevant property of the random walk is that a one-dimensional random walk is *re-current*; that is, with probability one it returns to the origin infinitely many times. Thus with probability one the balance-set  $Z(x)$  includes infinitely many balance points, so that with probability one the local level set  $L_x^{loc}$  has the structure of a Cantor set, hence is uncountable. This corresponds to a full Lebesgue measure set of points  $x$  having a local level set that is uncountable.

For a treatment of Hausdorff dimension, see Falconer [16]. To establish the Hausdorff dimension 0 assertion, we use the result that with probability one the number of times a simple random walk returns to the origin in the first  $n$  steps is  $o(n)$  as  $n \rightarrow \infty$ ; in fact with probability one it is  $O\left(n^{\frac{1}{2}+\epsilon}\right)$  as  $n \rightarrow \infty$ . (See Feller [17, Chap III], [18, Chap XII]). The proof is then completed by the following deterministic result.

**Claim.** *Any  $x = 0.b_1b_2b_3\dots$  that has the property that the number of returns to the origin in the first  $n$  steps of the corresponding random walk is  $o(n)$  as  $n \rightarrow \infty$  necessarily has Hausdorff dimension  $\dim_H(L_x^{loc}) = 0$ .*

To show this, let the balance points of  $Z(x)$  be  $0 < c_1 < c_2 < c_3 < \dots$ . The hypothesis on  $x$  implies

$$\lim_{k \rightarrow \infty} \frac{k}{c_k} = 0.$$

We can now cover  $L_x^{loc}$  by  $2^k$ -dyadic intervals of length  $2^{-c_k}$ , since there are only  $2^k$  possible flipped sequences; call this covering  $\mathcal{C}_k$ . For any  $\delta > 0$ , we have

$$\lim_{k \rightarrow \infty} \sum_{I_j \in \mathcal{C}_k} |I_j|^\delta = 2^k 2^{-\delta c_k} = 0,$$

since  $\frac{c_k}{k} \rightarrow \infty$ . This proves the claim, which completes the proof.  $\square$

### 3.3. Local level sets containing rational numbers

Knuth [23, Sect. 7.2.1.3, Exercise 83] raised the question of determining which rational  $y$  have an uncountable level set  $L(y)$ . We address here the easier question of determining which rational numbers  $x$  have an uncountable local level set  $L_x^{loc}$ . We also show that uncountable local level sets that contain a rational necessarily have positive Hausdorff dimension.

**Theorem 3.3.** (Rational local level sets)

*For a rational number  $x = \frac{p}{q} \in [0, 1]$ , the following properties are equivalent.*

- (1) *The local level set  $L_x^{loc}$  has positive Hausdorff dimension.*
- (2) *The local level set  $L_x^{loc}$  is uncountable.*
- (3) *The binary expansion of  $x$  has a purely periodic part with an equal number of zeros and ones, and also has a preperiodic part with an equal number of zeros and ones.*

*Moreover, if these equivalent properties hold, then  $\dim_H(L_x^{loc}) = \frac{k}{r}$ , where  $r$  is the number of bits in the periodic part of the binary expansion of  $x$  and  $k$  is the number of balance points per period.*

**Proof.** Trivially, (1) implies (2).

To show (2) implies (3), let  $x = 0.b_1b_2b_3\dots$  be the binary expansion of  $x = \frac{p}{q}$ . Let the balance-set  $Z(x) := \{c_j : D_{c_j}(x) = 0\}$  be the set of balance points of  $x$ , as defined in (1.3). By inspection, condition (3) is equivalent to the set  $Z(x)$  having infinite cardinality. From the definition of local level set  $L_x^{loc}$ , we see that the cardinality of  $L_x^{loc}$  is

$$\#L_x^{loc} = 2^{\#Z(x)}.$$

Hence,  $L_x^{loc}$  is uncountable if and only if (3) holds.

It remains to show that (3) implies (1). Since Hausdorff dimension is invariant under scaling, translation, and finite union, we may reduce to the case where the rational  $x$  has no preperiodic part; that is,  $x = 0.(b_1b_2\dots b_r)^\infty$ , with  $r$  the length of the periodic part. Say that these  $r$ -bits are partitioned into precisely  $1 \leq k \leq \frac{r}{2}$  blocks by the balance points of  $x$ . Let  $B_1, \dots, B_{2^k}$  be the  $2^k$  possible dyadic rationals obtained by applying block-flipping operations to  $B_1 = 0.b_1\dots b_r$ . Define the function  $S_i(x) := B_i + \frac{x}{2^r}$ . Using the terminology of Falconer [16, §9.2], the functions  $\{S_1, \dots, S_{2^k}\}$  form an iterated function system with attractor equal to the local level set  $L_x^{loc} = \bigcup_{i=1}^{2^k} S_i(L_x^{loc})$ . Furthermore, the open set  $V = (0, 1)$  is a bounded open set for which  $V \supseteq \bigcup_{i=1}^{2^k} S_i(V)$  with the union disjoint. Therefore, it satisfies the hypothesis of [16, Theorem 9.3], whose conclusion yields that the Hausdorff dimension  $\dim_H(L_x^{loc}) = s$  where  $s$  is the unique number for which  $\sum_{i=1}^{2^k} c_i^s = 1$ , and  $c_i = 2^{-r}$  is the ratio of similitude of the operator  $S_i$ . The equation is easily solved for  $s$  yielding  $s = \frac{k}{r} > 0$ . Thus, (3) implies (1). (It is also a consequence of [16, Theorem 9.3] that the  $s$ -dimensional Hausdorff measure of  $L_x^{loc}$  is a finite positive value.)  $\square$

**Remarks.** (1) By Theorem 3.3(3), any local level set of a rational number  $x$  that is uncountable necessarily contains infinitely many rational numbers  $x'$ , by using periodic sequences of flippings.

(2) A dyadic rational  $x$  always belongs to a finite local level set  $L_x^{loc}$ . This follows immediately from Theorem 3.3 since local level sets are either finite or uncountable.

## 4. Left Endpoints of Local Level Sets

We study the set of endpoints of local level sets. The leftmost endpoints fall in  $[0, \frac{1}{3}]$  and the rightmost endpoints in  $[\frac{2}{3}, 1]$ , and these sets are related by the operation  $x \rightarrow 1 - x$ . Therefore suffices to study the leftmost endpoint set, denoted  $\Omega^L$ . The usefulness of this set is that it parametrizes the complete collection of all local level sets, and the Takagi function turns out to be well-behaved when restricted to  $\Omega^L$ . Theorem 4.6 below describes the main properties of  $\Omega^L$ .

### 4.1. Deficient digit set

We make the following definition, which will be shown later to coincide with the set of all leftmost endpoints of local level sets.

**Definition 4.1.** The *deficient digit set*  $\Omega^L$  consists of all points

$$\Omega^L := \left\{ x = \sum_{j=1}^{\infty} \frac{b_j}{2^j} : D_j(x) \geq 0 \text{ for all } j \geq 1 \right\},$$

in which the deficient digit function  $D_j(x) = j - 2N_j^1(x)$  counts the number of binary digits 0's minus the number of 1's in the first  $n$  digits. Note that dyadic rationals  $\frac{m}{2^r}$  have two different binary expansions; at most one of these two expansions can belong to  $\Omega^L$ ; if one of them does, then by convention we assign the dyadic rational to the set  $\Omega^L$ .

We will establish in Theorem 4.6 that the deficient digit set  $\Omega^L$  is a Cantor set having Lebesgue measure zero. We need the following result as a preliminary step. Recall that a dyadic rational binary expansion  $x := \frac{k}{2^n} = 0.b_1b_2\dots b_n0^\infty$  (with  $k$  odd) is said to be *balanced* if  $D_n(x) = 0$ , and that  $n = 2m$  is necessarily even.

**Lemma 4.2.** (Balanced dyadic rationals in  $\Omega^L$ ) *For a fixed integer  $m \geq 0$ , the set  $\mathcal{B}'$  of dyadic rationals  $\frac{k}{2^{2m}} = 0.b_1b_2\dots b_{2m}$  that are balanced and belong to  $\Omega^L$ , i.e. have digit sums satisfying*

$$D_j\left(\frac{k}{2^{2m}}\right) \geq 0 \text{ for } 1 \leq j \leq 2m-1 \quad \text{and} \quad D_{2m}\left(\frac{k}{2^{2m}}\right) = 0,$$

*has cardinality the  $m$ -th Catalan number  $C_m = \frac{1}{m+1} \binom{2m}{m}$ , with  $C_0 = 1$ .*

**Proof.** Each such dyadic rational describes a lattice path starting from  $(0,0)$  and taking steps  $(1,1)$  or  $(1,-1)$  in such a way as to stay on or above the line  $y = 0$ . Here the  $j$ -th step  $(1,1)$  corresponds to  $b_j = 0$  and  $(1,-1)$  to  $b_j = 1$ ; the last step necessarily has  $b_{2m} = 1$ . Such steps can be counted using Bertrand's ballot theorem (Feller [17, p.73]). To apply the theorem we count paths from  $(0,0)$  to  $(n,x) = (2m+1,1)$  that stay strictly above the  $x$ -axis. Note that all such paths must go through  $(1,1)$ , and that to end at  $(2m+1,1)$  the last step must be  $(1,-1)$ . The number of paths is therefore

$$\frac{1}{2m+1} \binom{2m+1}{m} = \frac{1}{m+1} \binom{2m}{m} = C_m,$$

as asserted.  $\square$

We will determine the set of open intervals removed from  $[0,1]$  to create the deficient digit sum set  $\Omega^L$ . The following set  $\mathcal{B}$  supplies labels for these open intervals.

**Definition 4.3.** (1) The *breakpoint set*  $\mathcal{B}'$  is the set of all balanced dyadic rationals in  $\Omega^L$ . It consists of  $B'_0 = 0$  together with the collection of all dyadic rationals  $B' = \frac{n}{2^{2m}}$  that have binary expansions of the form

$$B' = 0.b_1b_2\dots b_{2m-1}b_{2m} \quad \text{for some } m \geq 1,$$

that satisfy the condition

$$D_j(B') \geq 0 \quad \text{for } 1 \leq j \leq 2m-1, \quad \text{and} \quad D_{2m}(B') = 0.$$

(2) The *small breakpoint set*  $\mathcal{B}$  is the subset of the breakpoint set  $\mathcal{B}'$  consisting of  $B_\emptyset = 0$  plus all members of  $\mathcal{B}'$  satisfying the extra condition that the last two binary digits  $b_{2m-1} = b_{2m} = 1$ . We may rewrite a dyadic rational in the small breakpoint set as

$$B = 0.b_1b_2\dots b_\ell 01^k, \quad \text{with } k \geq 2, \quad (4.1)$$

where  $2m = k + \ell + 1$ .

We show that values in the small breakpoint set  $\mathcal{B}$  naturally label the left endpoints  $x(B)^-$  of the intervals removed from  $[0, 1]$  to create the deficient digit set  $\Omega^L$ . (It is also possible to give a nice labeling for the right endpoints  $x(B)^+$  which we omit here.)

**Definition 4.4.** For each dyadic rational  $B = 0.b_1b_2\dots b_\ell 01^k$ ,  $k \geq 2$  in the small breakpoint set  $\mathcal{B}$  ( $B \neq B_\emptyset$ ) we associate the open interval

$$I_B := (x(B)^-, x(B)^+)$$

having the endpoints

$$\begin{aligned} x(B)^- &:= 0.b_1b_2\dots b_\ell 01^k(01)^\infty \\ x(B)^+ &:= 0.b_1b_2\dots b_\ell 10^k(00)^\infty. \end{aligned}$$

For  $B = B_\emptyset$  we set

$$I_{B_\emptyset} := (x(B_\emptyset)^-, x(B_\emptyset)^+) := (0.(01)^\infty, 1.(00)^\infty) = \left(\frac{1}{3}, 1\right).$$

Some data on  $I_B$  for small  $\ell$  and  $k$  appears in Table 1.

	$B = j/2^{2m}$	$x(B)^-$	$x(B)^+$	$\tau(x(B)^-)$	$\tau(x(B)^+)$
$2m = 4$					
	$3/16 = .0011$	$5/24 = .0011(01)^\infty$	$1/4 = .0100$	$13/24$	$1/2$
$2m = 6$					
	$7/64 = .000111$	$11/96 = .000111(01)^\infty$	$1/8 = .001000$	$37/96$	$3/8$
	$11/64 = .001011$	$17/96 = .001011(01)^\infty$	$3/16 = .001100$	$49/96$	$1/2$
	$19/64 = .010011$	$29/96 = .010011(01)^\infty$	$5/16 = .010100$	$61/96$	$5/8$

Table 1: Binary expansions of  $B$ ,  $x(B)^-$ , and  $x(B)^+$  for  $B \in \mathcal{B}$  of the form  $B = \frac{j}{2^{2m}}$  and corresponding  $\tau(x(B)^-)$  and  $\tau(x(B)^+)$ .

**Lemma 4.5.** *The open intervals  $\{I_B : B \in \mathcal{B}\}$  have the following properties.*

(1) *The intervals  $I_B$  for  $B \in \mathcal{B}$  are all disjoint from the deficient digit set  $\Omega^L$  and from each other. All the endpoints  $x(B)^\pm$  belong to  $\Omega^L$ , with the exception of  $x(B_\emptyset)^+ = 1$ .*

(2) *For  $B$  in the small breakpoint set  $\mathcal{B}$  there holds*

$$x(B)^+ - x(B)^- = \tau(x(B)^-) - \tau(x(B)^+) = \frac{1}{2^{k+\ell} \cdot 3} = \frac{1}{2^{2m-1} \cdot 3}$$

*so that  $x(B)^+ > x(B)^-$ . Thus the ratio  $\frac{\tau(x(B)^+) - \tau(x(B)^-)}{x(B)^+ - x(B)^-} = -1$ .*



**Proof.** (1) We have  $2m = k + \ell + 1$  in (4.1), and with the definition of  $x(B)^+, x(B)^-$  this implies that there are odd integers  $n_1, n_2$  such that

$$\begin{aligned} x(B)^- &= \frac{n_1}{2^{k+\ell+1}} + \frac{1}{2^{k+\ell+1} \cdot 3} \\ x(B)^+ &= \frac{n_2}{2^{\ell+1}}. \end{aligned}$$

Furthermore one easily sees that  $x(B)^+ > x(B)^-$  and that

$$x(B)^+ - x(B)^- = \frac{2}{2^{k+\ell+1} \cdot 3} = \frac{1}{2^{2m-1} \cdot 3}. \quad (4.2)$$

A key property of  $x(B)^-$  is that it belongs to  $\Omega^L$  and satisfies

$$D_{k+\ell+1+2j}(x(B)^-) = 0 \quad \text{for all } j \geq 0.$$

Now the binary expansion of any number  $x$  strictly between  $x(B)^-$  and  $x(B)^+$  first differs from  $x(B)^-$  in a bit at location  $\ell' > \ell$ , with  $x$  having digit 1 and  $x(B)^-$  digit 0. But a 1 in digit location  $k + \ell + 2j$ ,  $j \geq 1$  would then produce

$$D_{k+\ell+2j}(x) = -1,$$

which certifies  $x \notin \Omega^L$ . If there is instead a change from 0 to 1 in a digit of  $x(B)^-$  in location  $j \leq \ell + 1$ , then this would make  $x \geq x(B)^+$ , a contradiction. We conclude that no point of the open interval  $I_B$  belongs to  $\Omega^L$ .

We note by inspection that all upper endpoints  $x(B)^+ \in \Omega^L$ , except  $x(B_\emptyset)^+ = 1$ . This certifies that all open intervals  $I_B$  are disjoint (except possibly  $I_{B_\emptyset}$ ), because the endpoints of the closure of each such interval are in  $\Omega^L$  but the interiors are not. (This prevents both overlap and inclusion.) Finally,  $I_B \subset (0, \frac{1}{3})$  for each interval with  $B \neq B_\emptyset$ , yielding disjointness in all cases.

(2) In view of (4.2) it remains to verify that

$$\tau(x(B)^-) - \tau(x(B)^+) = \frac{1}{2^{k+\ell} \cdot 3}. \quad (4.3)$$

In the notation of Lemma 2.1 we write

$$y(B)^+ := \tau(x(B)^+) = \sum_{j=1}^{\infty} \frac{\ell_j^+}{2^j}; \quad y(B)^- := \tau(x(B)^-) = \sum_{j=1}^{\infty} \frac{\ell_j^-}{2^j}.$$

Now  $\ell_j^- = \ell_j^+$  for  $1 \leq j \leq \ell$ , since the two expansions agree. We also have  $\ell = 2m - k - 1$  and find that for  $B \in \mathcal{B}$  the first  $\ell$  digits of  $B$  necessarily contain  $m_1 = m - k = \frac{\ell-k+1}{2}$  values  $b_j = 1$  and  $\frac{\ell+k-1}{2}$  values  $b_j = 0$ . Now we calculate using Lemma 2.1 that

$$\ell_{\ell+1}^- = \frac{\ell - k + 1}{2}, \text{ and } \ell_{\ell+1}^+ = \frac{\ell + k - 1}{2}$$

while for  $2 \leq j \leq k + 1$  we have

$$\ell_{\ell+j}^- = \frac{\ell + k + 1}{2}, \text{ and } \ell_{\ell+j}^+ = \frac{\ell - k + 3}{2}.$$

Thus in the first  $2m = k + \ell + 1$  terms we have

$$\sum_{j=1}^{k+\ell+1} \frac{\ell_j^- - \ell_j^+}{2^j} = -\left(\frac{k-1}{2^{\ell+1}}\right) + \sum_{j=1}^k \frac{k-1}{2^{\ell+1+j}} = \frac{k-1}{2^\ell} \left(-\frac{1}{2} + \sum_{j=2}^{k+1} \frac{1}{2^j}\right) = -\frac{k-1}{2^{k+\ell+1}}. \quad (4.4)$$

For the terms  $j \geq k + \ell + 1$ , for  $x(B)^+$  we have  $(m - k + 1)$  1's in the first  $2m = k + \ell + 1$  positions, hence

$$\sum_{j=2m+1}^{\infty} \frac{\ell_j^+}{2^j} = \sum_{j=2m+1}^{\infty} \frac{m - k + 1}{2^j} = \frac{m - k + 1}{2^{k+\ell+1}}. \quad (4.5)$$

For  $x(B)^-$ , we have equal numbers of 0's and 1's at the  $2m$ -th digit, so that  $\ell_j^- = 2m + \tilde{\ell}_j$  where  $\tilde{\ell}_j$  correspond to the expansion of  $\frac{1}{3} = 0.(01)^\infty$ . Since we have  $\tau(\frac{1}{3}) = \frac{2}{3}$ , we obtain

$$\sum_{j=2m+1}^{\infty} \frac{\ell_j^-}{2^j} = \sum_{j=2m+1}^{\infty} \frac{m}{2^j} + \sum_{j=2m+1}^{\infty} \frac{\tilde{\ell}_j}{2^j} = \frac{m}{2^{2m}} + \frac{2}{3} \cdot \frac{1}{2^{2m}} = \frac{m + \frac{2}{3}}{2^{\ell+k+1}}. \quad (4.6)$$

Combining (4.4)-(4.6) yields

$$\tau(x(B)^-) - \tau(x(B)^+) = \frac{1}{2^{k+\ell+1}} \left( -(k-1) + \left(m + \frac{2}{3}\right) - (m - k + 1) \right) = \frac{2}{3} \cdot \frac{1}{2^{\ell+k+1}} = \frac{1}{2^{k+\ell} \cdot 3},$$

verifying (4.3).  $\square$

## 4.2. Properties of the deficient digit set

The following result characterizes the deficient digit set  $\Omega^L$ .

**Theorem 4.6.** (Properties of the deficient digit set)

(1) The deficient digit set  $\Omega^L$  comprises the set of leftmost endpoints of all local level sets. It satisfies  $\Omega^L \subset [0, \frac{1}{3}]$ .

(2) The deficient digit set  $\Omega^L$  is a closed, perfect set (Cantor set). It is given by

$$\Omega^L = [0, 1) \setminus \bigcup_{B \in \mathcal{B}} I_B, \quad (4.7)$$

where the omitted open intervals  $I_B$  have right endpoint a dyadic rational and left endpoint a rational number with denominator  $3 \cdot 2^k$  for some  $k \geq 1$ .

(3) The deficient digit set  $\Omega^L$  has Lebesgue measure zero.

**Proof.** (1) This property is immediate from the definition of local level set. The leftmost endpoint of any local level set satisfies  $D_j(x) \geq 0$  for all  $j \geq 1$  and is the only point in  $L_x^{loc}$  with this property.

(2) The definition of  $\Omega^L$  shows that it is a closed set, since the inequalities  $D_j(x) \geq 0$  are preserved under pointwise limits. Note here that any infinite binary expansion ending .11111... is excluded from membership on  $\Omega^L$ . To see that  $\Omega^L$  is a perfect set, we show each member of  $\Omega^L$  is a limit point of other members of  $\Omega^L$ . For each member  $x \in \Omega^L$  whose binary expansion contains an infinite number of 1's, we approximate it from below by the sequence

$x_n = \frac{1}{2^n} \lfloor 2^n x \rfloor \in \Omega^L$  obtained by truncating it at the  $n$ -th digit. For a dyadic rational member  $x = \frac{k}{2^j}$ , which necessarily ends in an infinite string of zeros, we approximate it from above using the sequence  $x_n = x + \frac{1}{2^{n+j+2}} \in \Omega^L$ .

To show the equality (4.7) set  $\Omega_C^L := [0, 1) \setminus \bigcup_{B \in \mathcal{B}} I_B$ . We clearly have  $\Omega^L \subseteq \Omega_C^L$ , by Lemma 4.5(1). It remains to show  $\Omega_C^L \subseteq \Omega^L$ . We check the contrapositive, that  $x \notin \Omega^L$  implies  $x \notin \Omega_C^L$ . We use the criterion that if  $x \notin \Omega^L$  then  $D_j(x) < 0$  for some  $j \geq 1$ . Now one can verify that the removed intervals  $I_B$  each detect those  $x$  whose first occurrence of  $D_j(x) < 0$  is in a specified digit position  $j$ , with a specified digit pattern of the first  $k$  digits, followed by some string of  $(01)^r$ , and these enumerate all possibilities of this kind. Thus  $x \notin \Omega_C^L$ , showing (4.7). The properties of the endpoints of  $I_B$  are given in Lemma 4.5(2).

(3) The set  $\Omega^L$  is shown to have Lebesgue measure 0 by covering it with dyadic boxes at level  $2m$  each of size  $\frac{1}{2^{2m}}$ , and noting from Lemma 4.2 that exactly  $C_m$  such boxes need to be used to cover  $\Omega^L$ , so that

$$\text{meas}(\Omega^L) \leq \frac{C_m}{2^{2m}}.$$

Stirling's formula gives for the Catalan numbers  $C_m$  the estimate

$$C_m = (1 + o(1)) \frac{1}{\sqrt{\pi m^3}} 2^{2m}, \quad \text{as } m \rightarrow \infty.$$

From this we see that  $\frac{C_m}{2^{2m}} \rightarrow 0$  as  $m \rightarrow \infty$ .  $\square$ .

### 4.3. Takagi function on deficient digit set

We next consider the Takagi function restricted to the deficient digit set  $\Omega^L$ . We show firstly that it has a weak increasing property approaching any point of  $\Omega^L$ , and secondly that it is nondecreasing when further restricted to the set  $\frac{1}{2}\Omega^L := \{\frac{1}{2}x : x \in \Omega^L\}$ . Note the characterization

$$\frac{1}{2}\Omega^L = \{x \in [0, 1] : D_j(x) > 0 \text{ for all } j \geq 1\}, \quad (4.8)$$

which shows that  $\frac{1}{2}\Omega^L$  is a subset of  $\Omega^L$ .

The following weak increasing property will be used in the proof of Theorem 1.5 to describe the points of increase of the Takagi singular function, and in the proof of Theorem 6.3 determining the expected number of local level sets per level.

**Theorem 4.7.** *Let  $x$  belong to the deficient digit set  $\Omega^L$ .*

(1) *If  $x$  has a binary expansion that does not end in  $0^\infty$ , then there exists a strictly increasing sequence  $\{x_k\}_{k=1}^\infty \subset \Omega^L$  such that*

$$\lim_{k \rightarrow \infty} x_k = x \quad \text{and all } \tau(x_k) < \tau(x).$$

(2) *If  $x$  has a binary expansion that does not end in  $(01)^\infty$ , then there exists a strictly decreasing sequence  $\{x_k\}_{k=1}^\infty \subset \Omega^L$  such that*

$$\lim_{k \rightarrow \infty} x_k = x \quad \text{and all } \tau(x_k) > \tau(x).$$

(3) *If  $x \in \frac{1}{2}\Omega^L$ , then (1), (2) above hold with the stronger property that all  $\{x_k\}_{k=1}^\infty \subset \frac{1}{2}\Omega^L$ .*

**Proof.** (1) The condition that the binary expansion of  $x \in \Omega^L$  not end in  $0^\infty$  is necessary for there to exist an infinite sequence  $x_1 < x_2 < x_3 < \dots \subset \Omega^L$  such that  $\lim_{k \rightarrow \infty} x_k = x$ .

Write  $x = 0.b_1b_2b_3\dots$ , and note that there must be infinitely many indices  $m_1 < m_2 < \dots$  with  $D_{m_k}(x) > 0$ . We then choose the  $x_k$  to be the dyadic rationals obtained by suitably truncating the binary expansion of  $x$  at these points:

$$x_k := 0.b_1b_2\dots b_{m_k}0^\infty. \quad (4.9)$$

We clearly have  $x_k \in \Omega^L$  and  $\lim_{k \rightarrow \infty} x_k = x$ , and  $x_1 \leq x_2 \leq x_3 < \dots$ . This sequence contains an infinite strictly increasing subsequence since the binary expansion of  $x$  does not end in  $0^\infty$ . We also note that if  $x \in \frac{1}{2}\Omega^L$ , i.e. if each  $D_j(x) \geq 1$ , then each  $x_k \in \frac{1}{2}\Omega^L$ .

It remains to show that  $\tau(x_k) < \tau(x)$ . By Lemma 2.1 we have

$$\tau(x) = \sum_{j=1}^{\infty} \frac{\ell_j}{2^j}. \quad (4.10)$$

Letting  $N^1(j) := N_j^1(x)$  (resp.  $N^0(j) := N_j^0(x)$ ) count the number of 1's (resp. 0's) in the first  $j$  digits of the binary expansion of  $x$ , we have

$$\tau(x_k) = \sum_{j=1}^{m_k} \frac{\ell_j}{2^j} + \sum_{j=m_k+1}^{\infty} \frac{N^1(m_k)}{2^j}. \quad (4.11)$$

Now for all  $j > m_k$  we have

$$N^1(m_k) = \min(N^0(m_k), N^1(m_k)) \leq \min(N^0(j-1), N^1(j-1)) \leq \ell_j,$$

and strict inequality holds here for at least one  $j > m_k$  because the binary expansion of  $x$  does not end in  $0^\infty$ . We conclude that  $\tau(x_k) < \tau(x)$  by comparing (4.10) and (4.11) term by term.

(2) The condition that the binary expansion of  $x \in \Omega^L$  not end in  $(01)^\infty$  is necessary for there to exist an infinite sequence  $x_1 > x_2 > x_3 > \dots \subset \Omega^L$  such that  $\lim_{k \rightarrow \infty} x_k = x$ .

Writing  $x = 0.b_1b_2b_3\dots$ , there must be infinitely many indices  $m_1 < m_2 < m_3 < \dots$  such that

$$D_j(x) \geq D_{m_k}(x) \quad \text{for all } j \geq m_k. \quad (4.12)$$

We now choose  $x_k$  to be the rational numbers:

$$x_k := 0.b_1b_2\dots b_{m_k}(01)^\infty. \quad (4.13)$$

We clearly have  $x_k \in \Omega^L$  and  $\lim_{k \rightarrow \infty} x_k = x$ , and we have  $x_1 \geq x_2 \geq x_3 \geq \dots$  using the fact that  $x \in \Omega^L$  together with (4.12), which implies that  $D_j(x_{k+1}) \geq D_j(x_k)$  for all  $j \geq 1$ . This sequence contains an infinite strictly decreasing subsequence since the binary expansion of  $x$  does not end in  $(01)^\infty$ . Again note that if  $x \in \frac{1}{2}\Omega^L$ , so all  $D_j(x) \geq 1$ , then all  $x_k \in \frac{1}{2}\Omega^L$ .

It remains to show that  $\tau(x_k) > \tau(x)$ . For any  $j \geq 1$ , set  $x[j] := 0.b_{j+1}b_{j+2}\dots$  and note that all  $x[m_k] \in \Omega^L$  by virtue of condition (4.12). Similarly define  $x_k[j]$  and note that  $x_k[m_k] = 0.(01)^\infty \in \Omega^L$ . There are two cases.

*Case (i).* If  $D(m_k) = 0$  then  $m_k = 2m$  and

$$\tau(x) = \sum_{j=1}^{2m} \frac{\ell_j}{2^j} + \frac{m}{2^{m_k}} + \frac{1}{2^{m_k}} \tau(x[m_k])$$

while

$$\tau(x_k) = \sum_{j=1}^{2m} \frac{\ell_j}{2^j} + \frac{m}{2^{m_k}} + \frac{1}{2^{m_k}} \cdot \frac{2}{3}.$$

Since  $\tau(x) \leq \frac{2}{3}$  and the only  $x \in \Omega^L$  with  $\tau(x) = \frac{2}{3}$  is  $x = \frac{1}{3} = 0.(01)^\infty$ , we conclude that the strict inequality  $\tau(x_k) > \tau(x)$  holds.

*Case (ii).* If  $D(m_k) \geq 1$ , then since the first  $m_k + 1$  digits of  $x$  and  $x_k$  match we have, using Lemma 2.1,

$$\tau(x) = \sum_{j=1}^{m_k} \frac{\ell_j}{2^j} + \frac{N^1(m_k)}{2^{m_k}} + \frac{D_{m_k}(x)}{2^{m_k}} x[m_k] + \frac{1}{2^{m_k}} \tau(x[m_k]).$$

and

$$\tau(x_k) = \sum_{j=1}^{m_k} \frac{\ell_j}{2^j} + \frac{N^1(m_k)}{2^{m_k}} + \frac{D_{m_k}(x)}{2^{m_k}} x_k[m_k] + \frac{1}{2^{m_k}} \tau(x_k[m_k]).$$

Now  $x_k[m_k] = 0.(01)^\infty = \frac{1}{3} > x[m_k]$  and  $\tau(x_k[m_k]) = \frac{2}{3} \geq \tau(x[m_k])$ , so we conclude the strict inequality  $\tau(x_k) > \tau(x)$ , as required.

(3) Suppose  $x \in \frac{1}{2}\Omega^L$ . For (1) any truncation  $x_k$  given by (4.9) will automatically satisfy the defining property (4.8) for membership in  $\frac{1}{2}\Omega^L$ . Similarly for (2) any value  $x_k$  given by (4.13) will automatically satisfy (4.8).  $\square$

Next consider the Takagi function restricted to the set  $\frac{1}{2}\Omega^L$ . We show that the Takagi function is nondecreasing on this set, and moreover is strictly increasing off a certain specific countable set of  $x$ . We thank P. Allaart for the following proof to establish the nondecreasing property, which replaces our original argument.

**Theorem 4.8.** (1) *The Takagi function is nondecreasing on the set  $\frac{1}{2}\Omega^L$ .*

(2) *The Takagi function is strictly increasing on  $\frac{1}{2}\Omega^L$  away from a countable set of points, which are a subset of those rationals having binary expansions ending in  $0^\infty$  or  $(01)^\infty$ . For each level  $y$  the equation  $y = \tau(x)$  has at most two solutions with  $x \in \frac{1}{2}\Omega^L$ . Thus if  $x_1 < x_2 < x_3$  are all in  $\frac{1}{2}\Omega^L$  then  $\tau(x_3) > \tau(x_1)$ .*

**Proof.** (1) [Allaart] Let  $x < x' \in \frac{1}{2}\Omega^L$  have the binary expansions  $x = 0.b_1b_2\dots$  and  $x' = 0.b'_1b'_2\dots$ , and let  $n$  be the index of the first bit that differs in the two expansions: that is,  $b_i = b'_i$  for  $i < n$  and  $b_n = 0$ ,  $b'_n = 1$ . Now set  $x_1 := 0.b'_1\dots b'_n = \frac{k+1}{2^n}$ , for some  $k \geq 0$ . Clearly,  $x < x_1 \leq x'$ . Furthermore,  $D_i(x_1) = D_i(x') > 0$  for  $1 \leq i \leq n$ , while  $D_i(x_1) = D_n(x_1) + (i - n) > 0$  for all  $i > n$ , so that  $x_1 \in \frac{1}{2}\Omega^L$ . Lemma 2.4 applied to the interval  $[\frac{k+1}{2^n}, \frac{k+2}{2^n}]$  gives  $\tau_n(x_1) \leq \tau_n(x')$  since  $D_n(x_1) = D_n(x') > 0$ ; hence

$$\tau(x_1) = \tau_n(x_1) \leq \tau_n(x') \leq \tau(x').$$

Therefore to prove the nondecreasing property, it is enough to prove  $\tau(x) \leq \tau(x_1)$ . We prove the following stronger claim (which does not require that either  $x$  or  $x_1 \in \frac{1}{2}\Omega^L$ ).

**Claim.** If  $\frac{k}{2^n} \leq x < \frac{k+1}{2^n}$  and  $D_i(x) > 0$  for all  $i \geq n$ , then  $\tau(x) \leq \tau(\frac{k+1}{2^n})$ .

**Proof of claim.** The result is proved by induction on the value of  $m = D_n(x)$ , at each step proving it for all  $n \geq m$ . We use the self-affine formula of Lemma 2.5, on  $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ . Setting  $x_0 = \frac{k}{2^n}$ , for  $x = x_0 + \frac{w}{2^n}$  with  $0 \leq w \leq 1$ , it gives  $\tau(x) = \tau(x_0) + \frac{1}{2^n}(\tau(w) + D_n(x_0)w)$ . Taking  $w = 1$  gives  $\tau(\frac{k+1}{2^n}) = \tau(x_0) + \frac{1}{2^n}D_n(x_0)$ , and differencing yields

$$\tau(\frac{k+1}{2^n}) - \tau(x) = \frac{1}{2^n}(D_n(x_0) - (\tau(w) + D_n(x_0)w)). \quad (4.14)$$

Here we note that

$$D_i(w) = D_{n+i}(x) - D_n(x_0), \quad i \geq 1. \quad (4.15)$$

That is, the function  $D(\cdot)$  itself undergoes a linear shift under the variable change from  $x$  to  $w$ .

We begin the induction with the base case  $m = 1$ . We then have  $m = D_n(x) = D_n(x_0) = 1$ , so that for any  $n \geq 1$ , (4.14) becomes

$$\tau(\frac{k+1}{2^n}) - \tau(x) = \frac{1}{2^n}(1 - (\tau(w) + w)). \quad (4.16)$$

Using (4.15) the assumption  $D_i(x) > 0$  for  $i \geq n$  implies that  $D_i(w) \geq 0$  for all  $i \geq 1$ . This says that  $w \in \Omega^L$ , so we have  $w \leq \frac{1}{3}$  whence  $\tau(w) + w \leq \frac{2}{3} + \frac{1}{3} = 1$ . Substituting this inequality in (4.16) completes the base case.

For the inductive step, fix  $m \geq 1$  and assume the claim holds for all  $D_n(x) = m$  and all  $n \geq m$ . Now suppose  $D_n(x) = m + 1$ . We bisect the line segment  $[\frac{k}{2^n}, \frac{k+1}{2^n}] = [\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}) \cup [\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}]$  into the sections where the function  $f_n(x) := \tau_{n+1}(x) - \tau_n(x) = \frac{\langle\langle 2^n x \rangle\rangle}{2^n}$  has constant slopes  $+1$  and  $-1$  respectively and check the claim in these two cases:

*Case (i).* Suppose  $\frac{2k+1}{2^{n+1}} \leq x < \frac{k+1}{2^n}$ . Here  $D_{n+1}(x) = m$ , since  $D_j(x)$  is the slope of  $\tau_j(x)$  at  $x$  (see Lemma 2.4). The claim assumption gives  $D_i(x) > 0$  for all  $i \geq n + 1$ , so the induction hypothesis now applies to  $x$  at level  $n' = n + 1$ , to give  $\tau(x) \leq \tau(\frac{2k+2}{2^{n+1}}) = \tau(\frac{k+1}{2^n})$ .

*Case (ii).* Suppose  $\frac{k}{2^n} \leq x < \frac{2k+1}{2^{n+1}}$ . Now (4.14) becomes

$$\tau(\frac{k+1}{2^n}) - \tau(x) = \frac{1}{2^n}(m + 1 - (\tau(w) + (m + 1)w)),$$

and the Case (ii) range of  $x$  implies  $0 \leq w \leq \frac{1}{2}$ . However for this range of  $w$  we have

$$\tau(w) + (m + 1)w \leq 1 + \frac{m + 1}{2} \leq m + 1.$$

Substituting this in the previous inequality gives  $\tau(x) \leq \tau(\frac{k+1}{2^n})$ . (Note: no conditions on  $D_i(x)$  are required in Case (ii).) This completes the induction step, and the claim follows.

(2) By Theorem 4.7(3) we have  $\tau(x_1) < \tau(x_2)$  for any two points  $x_1$  and  $x_2$  in  $\frac{1}{2}\Omega^L$  with  $x_1 < x_2$  such that neither  $x_1$  nor  $x_2$  is a rational number with binary expansion ending in either  $0^\infty$  or  $(01)^\infty$ . More is true; the conditions in Theorem 4.7 imply furthermore that equality  $t = \tau(x_1) = \tau(x_2)$  for  $x_1 < x_2$  in  $\frac{1}{2}\Omega^L$  can occur only if  $x_1$  ends in  $(01)^\infty$  and  $x_2$  ends in  $0^\infty$ . Thirdly, using the nondecreasing property of  $\tau(x)$  on  $\frac{1}{2}\Omega^L$ , we infer that for any level  $y$  the equation  $y = \tau(x)$  has at most two solutions  $x \in \frac{1}{2}\Omega^L$ . (A countable set of values  $y$  having two solutions in  $\frac{1}{2}\Omega^L$  exists, with solutions being pairs  $(\frac{1}{2}x(B)^-, \frac{1}{2}x(B)^+)$  associated to all intervals  $I(B), B \in \mathcal{B}$ .)  $\square$

## 5. Takagi Singular Function

We now study the behavior of the Takagi function restricted to the left hand endpoints of all local level sets. This leads to defining a singular function whose points of increase are confined to these endpoints, which we name the *Takagi singular function*.

### 5.1. Flattened Takagi function and Takagi singular function

We consider the Takagi function restricted to the set  $\Omega^L$  and linearly interpolate it across all intervals removed in constructing  $\Omega^L$ , obtaining a new function, the flattened Takagi function, as follows.

**Definition 5.1.** The  $\Omega^L$ -projection function  $P^L(x) := x_b$  where  $x_b$  is the largest point  $x_b \in \Omega^L$  having  $x_b \leq x$ . This function is well-defined since  $\Omega^L$  is a closed set, and  $0 \in \Omega^L$ . It clearly has the projection property

$$P^L(P^L(x)) = P^L(x).$$

To compute  $P^L(x)$ , if  $x = 0.b_0b_1\dots$  has  $x \notin \Omega^L$  then we have

$$x_b = 0.b_0b_1\dots b_n(01)^\infty,$$

where  $n$  is the smallest location in the binary expansion of  $x$  such that  $D_j(x) \geq 0$  for  $j \leq n$ , but  $D_{n+1}(x) < 0$ . If no such  $n$  exists then  $x_b = x \in \Omega^L$ .

**Definition 5.2.** The *flattened Takagi function*  $\tau^L(x)$  is given by

$$\tau^L(x) := \tau(x_b) - (x - x_b) = \tau(P^L(x)) + P^L(x) - x.$$

This definition agrees with the Takagi function on the set  $\Omega^L$  of left hand endpoints of local level sets, and it is linear with slope  $-1$  across all omitted intervals  $I_B$  between such endpoints. According to Lemma 4.5(2) this function then linearly interpolates across those intervals, showing that  $\tau^L(x)$  is a continuous function. It captures all the variation of the Takagi function that is outside local level sets and “flattens it ” inside local level sets. It is pictured in Figure 2. (See also the data in Table 1.)

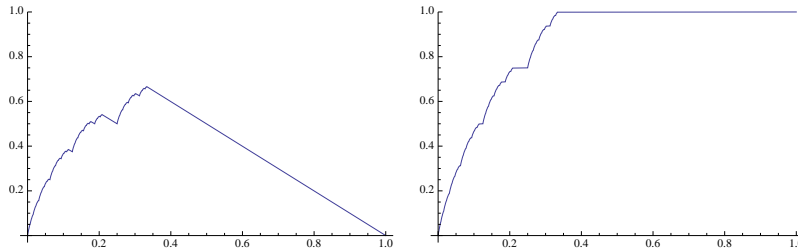


Figure 2: Graph of flattened Takagi function  $\tau^L(x)$  (left) and Takagi singular function  $\tau^S(x)$  (right) .

The flattened Takagi function has a countable set of intervals on which it has slope  $-1$ . It seems natural to adjust the function to have slope 0 on these intervals. Thus we study the following function.

**Definition 5.3.** The *Takagi singular function*  $\tau^S(x)$  is given by

$$\tau^S(x) := \tau^L(x) + x,$$

where  $\tau^L$  is the flattened Takagi function. That is,  $\tau^S(x) = \tau(x_b) + x_b$ .

It is also pictured in Figure 2.

We now prove Theorem 1.5 in the introduction, showing that  $\tau^S(x)$  is indeed a singular continuous function, justifying its name. Note that the statement of Theorem 1.5 gave a different definition of  $\tau^S$ , and part of the proof below is to show that this alternate definition coincides with the function given by Definition 5.3. The distributional derivative of  $\tau^S$  is a singular measure  $\mu_S$ , the Takagi singular measure, which we study in [25].

**Proof of Theorem 1.5.** We first show that the function  $\tau^S(x)$  given by Definition 5.3 is a monotone singular function. Lemma 4.5 (2) implies that the flattened Takagi function has slope  $-1$  across all omitted intervals outside of  $\Omega^L$ , hence the definition of the singular Takagi function given in Definition 5.3 guarantees that it is linear with slope 0 across all such intervals  $I_B$  for  $B \in \mathcal{B}$ . These intervals were shown to have full Lebesgue measure in Theorem 4.6 (3), so its variation is confined to the set  $\Omega^L$ , which is of measure 0. To conclude it is a Cantor function it remains to show that  $\tau^S(x)$  is a nondecreasing function on  $[0, 1]$ . Since it is constant away from  $\Omega^L$ , it suffices to show that  $\tau^S(x)$  is nondecreasing when restricted to  $\Omega^L$ . Now when  $x \in \Omega^L$  we have  $\tau^S(x) = \tau(x) + x$ , and by the dyadic self-similarity equation in Lemma 2.2,  $\tau(x) + x = 2\tau(\frac{x}{2})$ , and here  $\frac{x}{2} \in \frac{1}{2}\Omega^L$ . Therefore the nondecreasing property of  $\tau^S(x)$  is equivalent to showing that  $\tau(x)$  is nondecreasing when restricted to  $x \in \frac{1}{2}\Omega^L$ . This was shown in Theorem 4.8.

Next we show that the function  $\tau^S(x)$  given by Definition 5.3 coincides with the function defined in the theorem statement. That is, it has the two properties: (1)  $\tau^S(x) = \tau(x) + x$  for all  $x \in \Omega^L$ ; and (2)  $\tau^S(x) = \sup\{\tau^S(x_1) : x_1 \leq x \text{ with } x_1 \in \Omega^L\}$ . The first property holds since for  $x \in \Omega^L$  one has  $x = x_b$  so  $\tau^S(x) = \tau(x_b) + x_b = \tau(x) + x$ . The second property holds since  $\tau^S(x)$  is now known to be nondecreasing, whence

$$\tau^S(x) = \tau(x_b) + x_b = \sup\{\tau^S(x_1) : x_1 \leq x \text{ with } x_1 \in \Omega^L\}.$$

Finally we verify that the set  $\Omega^L$  is the closure of the set of points of increase of  $\tau^S(x)$ . It follows from Theorem 4.7 that all points of  $\Omega^L$  are points of increase except for a countable set of  $x \in \Omega^L$  which are rational numbers whose binary expansion ends in  $0^\infty$  or  $(01)^\infty$ . Since  $\Omega^L$  is a perfect set, these rational numbers are limit points of elements of  $\Omega^L$  that are irrational, hence they fall in the closure of the set of points of increase of  $\Omega^L$ .  $\square$ .

**Remark.** An alternate proof of the monotone property of  $\tau^S(x)$  in Theorem 1.5 can be based on defining piecewise linear approximation functions  $\tau_n^L(x)$  and  $\tau_n^S(x)$  to  $\tau^L(x)$  and  $\tau^S(x)$ , respectively, in an obvious fashion. One can prove by induction that each  $\tau_n^S(x)$  is a nondecreasing function, using Lemma 2.5(1). The approximations  $\tau_n^S(x)$  approach  $\tau^S(x)$  pointwise from below, giving the result.

## 5.2. Jordan decomposition of flattened Takagi function

We prove that the flattened Takagi function is of bounded variation. Recall that a *function of bounded (pointwise) variation*  $f$  on  $U = (0, 1)$  is a (possibly discontinuous) function whose



total variation, denoted  $\text{Var } f$  or  $V_0^1(f)$ , given by

$$V_0^1(f) = \text{Var } f := \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| : 0 < x_0 < x_1 < \cdots < x_n < 1, n \geq 1 \right\}$$

is finite. We let  $BPV((0,1))$  denote the set of functions of bounded (pointwise) variation on the open interval  $(0,1)$ , following the notation of Leoni [26, Chap 2]. (In the literature this space is usually denoted  $BV(I)$ , but this notation leads to a conflict with the geometric measure theory notation in Section 6.) Any function of bounded variation has a *monotone decomposition* (or *Jordan decomposition*)

$$f = f_u + f_d$$

in which  $f_u$  is an upward monotone (i.e. non-decreasing) bounded function, possibly with jump discontinuities, and  $f_d$  is a downward monotone (i.e. non-increasing) bounded function. Such a decomposition is not unique. Conversely, any function having a Jordan decomposition is of bounded pointwise variation. A *minimal monotone decomposition* is one such that

$$V_0^1(f) = V_0^1(f_u) + V_0^1(f_d).$$

**Theorem 5.4.** (Jordan decomposition of flattened Takagi function) *The flattened Takagi function  $\tau^L(x)$  is of bounded (pointwise) variation, so is in  $BPV((0,1))$ . It has a minimal monotone decomposition given by*

$$\tau^L(x) = f_u(x) + f_d(x), \tag{5.1}$$

with downward part  $f_d(x) = -x$  and upward part  $f_u(x) = \tau^L(x) + x$  both being continuous functions. The upward part is a singular function whose points of increase are supported on the deficient digit set  $\Omega^L$ . The total variation of the flattened Takagi function is  $V_0^1(\tau^L) = 2$ , with  $V_0^1(f_u(x)) = V_0^1(f_d(x)) = 1$ .

**Proof.** (1) The decomposition (5.1) holds by definition of  $\tau^S(x)$ . By Theorem 1.5  $f_u(x)$  is non-decreasing and bounded, and is a monotone singular function supported on  $\Omega^L$ . Clearly  $f_d(x) = -x$  is non-increasing and bounded, thus  $\tau^L(x)$  is of bounded variation, hence (5.1) is a monotone decomposition.

(2) The minimality of the monotone decomposition (5.1) is a consequence of the fact that the function  $f_d$  is absolutely continuous with respect to Lebesgue measure, while by Theorem 1.5 the function  $f_u$  is singular with respect to Lebesgue measure. Thus  $V_0^1(f) = V_0^1(f_u) + V_0^1(f_d) = 1 + 1 = 2$ , as asserted.  $\square$

## 6. Expected number of local level sets

The results of the last section show that the Takagi function restricted to the set  $\Omega^L$  is well behaved, giving a function of bounded variation. We now use the coarea formula of geometric measure theory for BV-functions to determine the expected number of local level sets on a random level  $0 \leq y \leq \frac{2}{3}$ .

We formulate the coarea formula for the one-dimensional case, using the terminology of Leoni [26]; note that the full power of this formula lies in the  $n$ -dimensional case. For an open

set  $U$  of the real line, the bounded variation space in the sense of geometric measure theory  $BV(U)$  consists of all functions  $f \in L^1(U)$  for which there exists a finite signed Radon measure  $\mu$  on the Borel sets of  $U$  such that

$$\int_U f(x)\phi'(x)dx = - \int_U \phi(x)d\mu$$

for all test functions  $\phi(x) \in C_c^1(U)$ , where we let  $C_c^k(U)$  denote the set of  $k$ -continuously differentiable functions with compact support. This signed measure  $\mu$  is called the *weak derivative* of  $f(x)$  and is denoted  $Df$ .

The Jordan decomposition theorem for measures says that the finite Radon measure  $Df$  decomposes uniquely into the difference  $Df = Df^+ - Df^-$  of mutually singular nonnegative measures, both of which are finite ([26, Theorem B.72]). The associated *total variation measure*  $|Df|$  is  $|Df| = Df^+ + Df^-$ . (See Evans and Gariepy [15, Sect 5.1, Theorem 1] for the  $n$ -dimensional case.) The total variation of a function  $f$  in  $BV(U)$  is expressible using test functions as

$$|Df|(U) = V(f, U) := \sup \left\{ \int_U f(x)\phi'(x)dx : \phi(x) \in C_c^1(U; \mathbb{R}), \|\phi\| = \max_{x \in U}(|\phi(x)|) \leq 1 \right\}. \quad (6.1)$$

We define the *perimeter* of a Lebesgue measurable set  $E \subset U$ , denoted  $|\partial E|(U)$  or  $P(E, U)$ , to be the total variation of its characteristic function  $\chi_E$  in  $U$ , i.e.  $|\partial E|(U) = |D\chi_E|(U)$ . It follows from (6.1) that if  $E$  is an open set in  $U = (0, 1)$  consisting of a finite number of non-adjacent intervals (where non-adjacent means no two intervals have a common endpoint), then the perimeter  $|\partial E|(U)$  counts the number of endpoints of the intervals inside  $U$ . (The perimeter does not detect the endpoints of  $U$ , e.g. the perimeter of  $E = (0, \frac{1}{2})$  has  $|\partial E| = 1$ .) For general open sets  $E \subset U$ , which may have infinitely many open intervals, the value of the perimeter is more complicated. Two extreme cases are: (1) The complement  $E := U \setminus C$  of the middle-third Cantor set  $C$  has perimeter  $|\partial E|(U) = 0$ ; (2) An open set  $E$  having a partition  $E = \bigcup_{i=1}^{\infty} U_i$  in which each open interval  $U_i = (a_i, b_i)$  has an adjacent open interval  $J_i = (b_i, b_i + \epsilon_i)$  disjoint from  $E$  necessarily has perimeter  $|\partial E|(U) = +\infty$ .

Since the classical bounded variation space  $BPV(U)$  (c.f. Section 5.2) consists of functions while the geometric measure theory space  $BV(U)$  consists of equivalence classes of functions agreeing on sets of full Lebesgue measure, these are distinct spaces. However they are closely related, as follows (Leoni [26, Theorem 7.2]).

**Proposition 6.1.** (Relation of  $BPV(U)$  and  $BV(U)$ .)

Let  $U \subset \mathbb{R}$  be an open set. (1) If  $f : U \rightarrow \mathbb{R}$  is an integrable function belonging to  $BPV(U)$ , then its  $L^1$ -equivalence class belongs to  $BV(U)$ , and satisfies

$$\text{Var } f \geq |Df|(U).$$

(2) Conversely any  $f \in BV(U)$  has a right continuous representative function  $\bar{f}$  in its  $L^1$ -equivalence class that belongs to  $BPV(U)$ , and it satisfies

$$\text{Var } \bar{f} = |Df|(U).$$

The following is a one-dimensional version of the coarea formula for functions in  $BV(U)$ .

**Proposition 6.2.** (Coarea formula for BV functions) Let  $U = (0, 1)$ . If  $f \in BV(U)$ , there holds:

(1) The upper set

$$E_t := E_t(f) = \{x \in U : f(x) > t\}$$

has finite perimeter  $|\partial E_t|$  for all but a Lebesgue measure 0 set of  $t \in \mathbb{R}$ , and the mapping

$$t \mapsto |\partial E_t|(U), \quad t \in \mathbb{R}$$

is a Lebesgue measurable function.

(2) In addition the variation measure  $|Df|$  of  $f$  satisfies

$$|Df|(U) = \int_{-\infty}^{\infty} |\partial E_t|(U) dt.$$

(3) Conversely, if  $f \in L^1(U)$  and  $\int_{-\infty}^{\infty} |\partial E_t|(U) dt < \infty$ , then  $f \in BV(U)$ .

**Proof.** Versions of the coarea formula for functions in  $BV(U)$  for a given open set  $U$  in  $\mathbb{R}^n$  are proved in Evans and Gariepy [15, Theorem 1, Sec. 5.5] and Leoni [26, Theorem 13.25]. Here we specialize to the case  $U = (0, 1)$ .  $\square$

In our application, all relevant functions  $f \in BPV(U)$  for  $U = (0, 1)$  are continuous on  $[0, 1]$ , in which case by Proposition 6.1(2) we have  $|Df|(U) = \text{Var } f = V_0^1(f)$ .

We use the Coarea formula for BV functions to compute the expected number of local level sets at a random level over the range  $0 \leq y \leq \frac{2}{3}$ , with respect to Lebesgue measure.

**Theorem 6.3.** (Expected number of local level sets) *For a full Lebesgue measure set of ordinate points  $y \in [0, \frac{2}{3}]$  the number  $N^{\text{loc}}(y)$  of local level sets at level  $y$  is finite. Furthermore  $N^{\text{loc}}(y)$  is a Lebesgue measurable function satisfying*

$$\int_0^{\frac{2}{3}} N^{\text{loc}}(y) dy = 1. \quad (6.2)$$

*That is, the expected number of local level sets on a randomly drawn ordinate level  $y$  is  $\frac{3}{2}$ .*

**Proof.** Theorem 5.4 shows that the flattened Takagi function  $\tau^L(x)$  belongs to  $BPV(U)$ , for  $U = (0, 1)$ . Using Proposition 6.1(1) we may view it in  $BV(U)$  and apply the coarea formula for BV functions (Proposition 6.2), taking  $f = \tau^L$  and  $U = (0, 1)$  to obtain

$$\int_{-\infty}^{\infty} |\partial E_t(\tau^L)|(U) dt = |D\tau^L|(U), \quad (6.3)$$

in which  $|\partial E_t(\tau^L)|(U)$  is a Lebesgue measurable function of  $t$ . Since the function  $\tau^L(x)$  is continuous on  $[0, 1]$ , Proposition 6.1(2) applies to give

$$|D\tau^L|(U) = V_0^1(\tau^L) = 2.$$

Thus we obtain

$$\int_0^{\frac{2}{3}} |\partial E_t(\tau^L)|(U) dt = 2.$$

We now study to what extent the integrand  $|\partial E_t(\tau^L)|(U)$  detects endpoints of local level sets at level  $t$ . The function  $N^{\text{loc}}(t)$  takes nonnegative integer values or  $+\infty$ .

**Claim.** For irrational  $t$  with  $0 < t < \frac{2}{3}$ , there holds

$$2N^{loc}(t) = |\partial E_t(\tau^L)|(U).$$

In particular,  $N^{loc}(t)$  is a Lebesgue measurable function, because it differs from  $\frac{1}{2}|\partial E_t(\tau^L)|(U)$  on a set of measure zero.

To prove the claim, we note that for  $t > 0$  the upper set  $E_t := E_t(\tau^L) = \bigcup_i I_i$  is a finite or countable disjoint union of nonempty open intervals, which all lie strictly inside  $U = (0, 1)$ . Let  $I = (a, b)$  be one such interval. If  $a \notin \Omega^L$ , then the point  $(a, \tau^L(a))$  would lie in the interior of a line segment of slope  $-1$  in the graph  $\mathcal{G}(\tau^L)$ . This is impossible since a point in the interior of a line of slope  $-1$  in  $\mathcal{G}(\tau^L)$  will have  $\tau^L(a + \epsilon) = t - \epsilon < t$  for all small enough positive  $\epsilon$ , contradicting  $I \subseteq E_t$ . Hence  $a \in \Omega^L$ , and consequently  $a$  is the left hand endpoint of a local level set in the level set  $\tau(x) = t$ . We next assert that the point  $b$  must correspond to a point in the interior of a line segment of slope  $-1$  in the graph  $\mathcal{G}(\tau^L)$ . This holds because, first, it cannot correspond to an endpoint of such a segment because then it would be a rational number with expansion ending in  $0^\infty$ , contradicting  $\tau(b) = t$  being irrational, and second, it cannot be a member of  $\Omega^L$ , since otherwise Theorem 4.7 (1) would imply there exists an increasing sequence of values  $\{x_k\} \subseteq \Omega^L$  with  $x_k \rightarrow b$ , having  $\tau(x_k) < \tau(b)$ , which would contradict  $b$  being the right endpoint of an interval of an upper set. Thus we know  $b$  corresponds to a point in the interior of a segment of slope  $-1$  in  $\mathcal{G}(\tau^L)$ , hence there is some positive  $\epsilon$  depending on  $b$  such that the adjacent open interval  $(b, b + \epsilon)$  is not contained in  $E_t$ . We conclude immediately that if  $E_t$  is an infinite disjoint union of nonempty open intervals, then  $|\partial E_t|(U) = \infty$ . Moreover  $N^{loc}(t) = \infty$ , proving the claim in this case.

To finish, we may assume  $E_t$  is a finite disjoint union of nonempty open intervals  $E_t = \bigcup_{i=1}^n (a_i, b_i)$ , and hence its boundary  $\partial E_t := \bar{E}_t \setminus E_t = \{a_1, \dots, a_n, b_1, \dots, b_n\}$ . By the above, each  $a_i \in \Omega^L$ , but each  $b_j \notin \Omega^L$ , so there are gaps between each of these intervals, and we conclude the perimeter  $|\partial E_t|(U) = 2n$ . It remains to show  $N^{loc}(t) = n$ . The fact that each  $a_i \in \Omega^L$  implies  $N^{loc}(t) \geq n$ . Now let  $x \in \Omega^L$  be the left hand endpoint of any local level set in the level set  $\tau(x) = t$ . It suffices to show that  $x = a_i$  for some  $1 \leq i \leq n$  since then  $N^{loc}(t) \leq \#\{a_1, \dots, a_n\} = n$ . We know  $x$  is irrational because  $\tau(x) = t$  is irrational by hypothesis. Theorem 4.7(2) then applies to show there is a decreasing sequence  $\{x_k\} \subseteq \Omega^L$  with  $x_k \rightarrow x$  and  $\tau(x_k) > \tau(x) = t$ . The existence of the sequence  $x_k$  implies that  $x \in \partial E_t = \{a_1, \dots, a_n, b_1, \dots, b_n\}$  since  $x \notin E_t$  but is given as a limit from of decreasing values  $x_k \in E_t$ . By the above,  $x \neq b_j$  for any  $j$ , and therefore  $x = a_i$  for some  $1 \leq i \leq n$ , finishing the proof of the claim.

The claim together with (6.3) gives (6.2).  $\square$ .

**Remark.** Theorem 6.3 gives no information concerning the multiplicity of local level sets on those levels having an uncountable level set, because the set of such levels  $y$  has Lebesgue measure 0.

## 7. Levels Containing Infinitely Many Local Level Sets

In this section we show that there exists a dense set of levels in  $[0, \frac{2}{3}]$  that contain a countably infinite number of local level sets; this complements Theorem 6.3. We first show this holds for the particular level set  $L(\frac{1}{2})$ . The fact that this level set is countably infinite was previously noted by Knuth [23, Sec. 7.2.1.3, Problem 82e].

**Theorem 7.1.** (Countably infinite level set) *The level set  $L(\frac{1}{2})$  is countably infinite, with  $L(\frac{1}{2}) = \mathcal{L}_1 \cup \mathcal{L}_2$  and*

$$\mathcal{L}_1 := \left\{ x_k := \frac{1}{2} - \sum_{j=1}^k \left(\frac{1}{4}\right)^j : k = 0, 1, 2, \dots, \infty \right\},$$

with  $\mathcal{L}_2 = \{1 - x : x \in \mathcal{L}_1\}$ . It contains an infinite number of distinct local level sets. One has  $L_{x_\infty}^{loc} = \{\frac{1}{6}, \frac{5}{6}\} \subset L(\frac{1}{2})$  and  $\frac{1}{6} \leq x \leq \frac{5}{6}$  for all  $x \in L(\frac{1}{2})$ .

**Proof.** First we show that each  $x \in \mathcal{L}_1 \cup \mathcal{L}_2$  satisfies  $\tau(x) = 1/2$ . By Lemma 2.2(1), it suffices to consider  $x \in \mathcal{L}_1$ . Let  $x := x_k := \frac{1}{2} - \sum_{j=1}^k (\frac{1}{4})^j$  for  $0 \leq k \leq \infty$ . For  $k \geq 1$  the dyadic rational  $x_k$  has two binary expansions, the first expansion being  $x_k^+ = 0.0(01)^{k-1}1$  and the second being  $x_k^- = 0.0(01)^{k-1}01^\infty$ . Here the first binary expansion clearly has all  $D_j(x_k^+) \geq 0$  for  $j \geq 1$  which certifies that  $x_k^+ \in \Omega^L$ . By direct calculation  $\tau(x_0) = \tau(x_1) = \frac{1}{2}$ . Now the flip operation shows for  $k \geq 1$  that

$$x_k^- = 0.0(01)^{k-1}01^\infty = 0.(0(01)^{k-1}1)1^\infty \sim 0.0(01)^k 10^\infty = x_{k+1}^+.$$

Thus we deduce

$$\tau(x_k^+) = \tau(x_k^-) = \tau(x_{k+1}^+),$$

whence by induction on  $k \geq 1$  we conclude that  $\tau(x_k) = \tau(x_k^+) = \tau(x_k^-) = \frac{1}{2}$  for all finite  $k \geq 1$ , as asserted. Finally the case  $k = \infty$ , with  $x_\infty = 0.0(01)^\infty = \frac{1}{6}$  has  $\tau(x_\infty) = \lim_{k \rightarrow \infty} \tau(x_k) = \frac{1}{2}$  by continuity of the Takagi function.

Next we observe that the local level sets are  $L_{x_\infty}^{loc} = \{\frac{1}{6}, \frac{5}{6}\}$  and

$$L_{x_k^+}^{loc} = \{x_k^+, x_{k-1}^-, 1 - x_k^+, 1 - x_{k-1}^-\}, \quad \text{for } k \geq 1.$$

For example, when  $k = 1$  this is  $L_{x_1^+}^{loc} = \{(\frac{1}{4})^+, (\frac{1}{2})^-, (\frac{3}{4})^-, (\frac{1}{2})^+\}$ . This shows that  $L(\frac{1}{2})$  has infinitely many local level sets.

It remains to show that  $L(\frac{1}{2})$  contains no elements other than those in  $\mathcal{L}_1 \cup \mathcal{L}_2$ . For this, in view of the symmetry of  $\tau(x)$ , it suffices to prove two assertions:

- (1) If  $x < \frac{1}{6} = x_\infty$ , then  $\tau(x) < \frac{1}{2}$ .
- (2) For all  $k$ , if  $x_k < x < x_{k-1}$ , then  $\tau(x) > \frac{1}{2}$ .

Let  $x < \frac{1}{6}$ . Lemma 2.2(1) implies that  $\tau(x) = \frac{1}{2}\tau(2x) + x$ . Combining this with the inequality  $\tau(x) \leq \frac{2}{3}$  proves (1).

If  $x$  satisfies  $x_k < x < x_{k-1} = x_k + \frac{1}{2^{2k}}$  for  $k \geq 1$ , then  $x = x_k + \frac{x'}{2^{2k}}$  for  $0 < x' < 1$ . Since  $D_{2k}(x_k) = 0$ , then by Lemma 2.5,

$$\tau(x) = \tau(x_k) + \frac{\tau(x')}{2^{2k}} > \frac{1}{2}.$$

This proves (2).  $\square$

Now define

$$\Lambda_\infty^{loc} := \{y : L(y) \text{ contains infinitely many different local level sets}\}. \quad (7.1)$$

Theorem 7.1 above shows that  $y = \frac{1}{2} \in \Lambda_\infty^{loc}$ . Also, recall from Definition 4.3 that the breakpoint set  $\mathcal{B}'$  is the set of all balanced dyadic rationals  $B' \in \Omega^L$ .

**Theorem 7.2.** (Levels with an infinite number of local level sets)

- (1) The set  $\Lambda_\infty^{loc}$  has Lebesgue measure 0. It is not a closed set.
- (2) For each  $B' = 0.b_1b_2 \cdots b_{2m} \in \mathcal{B}'$ , the value

$$y_{B'} := \tau(B') + \frac{1}{2^{2m+1}}$$

is a dyadic rational, and  $L(y_{B'})$  contains infinitely many disjoint local level sets. Furthermore, there are infinitely many dyadic rationals  $x \in \Omega^L$  with  $y_{B'} = \tau(x)$ .

- (3) The set of levels

$$\Delta_\infty^{loc} := \left\{ y_{B'} = \tau(B') + \frac{1}{2^{2m+1}} : B' \in \mathcal{B}' \right\}$$

is dense in  $[0, \frac{2}{3}]$ . Since  $\Delta_\infty^{loc} \subseteq \Lambda_\infty^{loc}$ , the set  $\Lambda_\infty^{loc}$  is dense in  $[0, \frac{2}{3}]$ .

**Proof.** (1) This measure 0 property of  $\Lambda_\infty^{loc}$  follows immediately from the expected number of local level sets being finite (Theorem 6.3). The fact that this set is not a closed set will follow once property (3) is proved.

(2) For each balanced dyadic rational  $B' = 0.b_1b_2 \cdots b_{2m}$  in  $\Omega^L$  we consider for  $k \geq 1$  the infinite set of dyadic rationals

$$x_k(B') := 0.b_1b_2 \cdots b_{2m}0(01)^{k-1}1 = B' + \frac{1}{2^{2m}}x_k,$$

where  $x_k = 0.0(01)^{k-1}1$  has  $\tau(x_k) = \frac{1}{2}$  by Theorem 7.1. Using the self-affine scaling property in Lemma 2.5 we have

$$\tau(x_k(B')) = \tau(B') + \frac{1}{2^{2m}}\tau(x_k) = \tau(B') + \frac{1}{2^{2m+1}},$$

so all points  $\tau(x_k(B'))$  are on the same level  $y = y_{B'}$ , and  $y_{B'}$  is necessarily a dyadic rational number. Clearly each  $x_k(B') \in \Omega^L$ , so each determines a different local level set, establishing (2).

(3) It is easy to see that the set of balanced dyadic rationals  $B' = 0.b_1 \dots b_{2m}$  having  $D_j(B') \geq 0$  for all  $j \geq 0$  and  $D_{2m} = 0$  is dense inside the deficient digit set  $\Omega^L$ . Indeed, given any  $x = 0.b_1b_2 \dots \in \Omega^L$ , the approximation  $x_k = 0.b_1b_2 \dots b_k 1^{D_k(x)} 0^\infty$  is such a dyadic rational having  $|x - x_k| \leq 2^{-k}$ . Since there is at least one local level set on each level, we have  $\tau(\Omega^L) = [0, \frac{2}{3}]$ . Since the flattened Takagi function is continuous, we conclude that the values  $y_{B'} = \tau(B' + \frac{1}{2^{2m+1}})$  are dense in  $[0, \frac{2}{3}]$ , as asserted.  $\square$

**Remarks.** (1) The proof above shows the stronger result that if  $y \in \Lambda_\infty^{loc}$  then for every balanced dyadic rational  $B'$  that belongs to  $\Omega^L$ , one has  $y_{B'}^* := B' + \frac{y}{2^{2m}} \in \Lambda_\infty^{loc}$ .

(2) One can ask whether the equality  $\Delta_\infty^{loc} = \Lambda_\infty^{loc}$  might hold, or (weaker) whether  $\Lambda_\infty^{loc}$  is a countable set.

## 8. Further Questions

This investigation of the structure of local level sets of the Takagi function raises a number of questions for further work.

(1) Theorem 1.4 shows that an abscissa generic local level set is uncountable with probability one, with  $x$  drawn uniformly from  $[0, 1]$ . Can one determine the expected number of local level sets at a level  $L(\tau(x))$ , with  $x$  drawn uniformly from  $[0, 1]$ ?

We note that Theorem 1.6 sheds no light regarding this question. It seems to involve properties related to a new measure  $\nu$ , supported on  $\Omega^L$ , which is mutually singular to both Lebesgue measure and to the Takagi singular measure  $\mu_S$ , which we hope to discuss elsewhere.

(2) Theorem 1.6 shows that the expected number of local level sets at a given height  $y$  drawn uniformly in  $[0, \frac{2}{3}]$  is  $\frac{3}{2}$ . There is an associated probability distribution

$$\text{Prob}[N^{loc}(y) = k] := \frac{3}{2} \text{meas}[y : N^{loc}(y) = k],$$

whose mean value is  $\frac{3}{2}$ . Can one explicitly compute these probabilities in closed form?

(3) Can one explicitly determine the Hausdorff dimension of the local level set  $L_x^{loc}$  in terms of properties of the binary expansion of  $x$ ? In particular, to what extent does the balance set  $Z(x)$  determine the Hausdorff dimension of  $L_x^{loc}$ ?

For rational  $x$ , we have calculated the Hausdorff dimension of  $L_x^{loc}$  in the proof of Theorem 3.3. For general  $x$ , recall that a necessary condition for positive Hausdorff dimension given in the proof of Theorem 1.4 is that  $\limsup_{k \rightarrow \infty} \frac{k}{c_k} > 0$ ; this condition depends only on  $Z(x)$ .

(4) Theorem 3.3 characterizes those rationals  $x$  which have an uncountable local level set  $L^{loc}$  in terms of their binary expansions. Can one explicitly characterize (e.g. in terms of binary expansion) which rational levels  $y$  contain some rational  $x$  for which  $L_x^{loc}$  is uncountable?

This problem, which is a weaker version of one proposed by Knuth [23, Sect. 7.2.1.3, Exercise 83], may be difficult.

(5) The Fourier series of the Takagi function, viewed as a periodic function of period 1, is explicitly known in closed form. Can one explicitly find the Fourier series of the flattened Takagi function  $\tau^L(s)$  or the Takagi singular function  $\tau^S(x)$ ?

The structure and behavior of monotone singular functions, particularly including their Fourier transforms, is a topic of some interest, tracing back to work of Hartman and Kershner [20], Salem [30], [31], [32]. See Dovgoshey et al [14] for a detailed treatment of the Cantor function.

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